这是一个小 topic，参考书籍为 Linear Algebra Done Right，3rd.

Chapter 3 Linear maps

用线性映射 (linear map) 定义矩阵

再由映射的运算定义矩阵的运算.

Conventional approach：由"数表"定义矩阵，规定运算.

prerequisite：线性空间.

> **Notation:**
> $\mathbb{F}$: $\mathbb{R}$ or $\mathbb{C}$
> V, W denote vector spaces over $\mathbb{F}$.

$\|$ linear space

--------------- Start ---------------

Section 1. Linear maps (= linear transformation 线性变换)

Def A linear map from V to W is a function $T: V \to W$ with the following properties:

(a) additivity : $T(u+v) = Tu + Tv$ for all $u, v \in V$

(b) homogenity : $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and $v \in V$

Notation The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$

Some examples.

zero. $0 \in \mathcal{L}(V, W)$

the function that takes each element of V to the additive identity of W

$$0v = 0$$

$0 \in \mathcal{L}(V, W)$ $\quad$ $0 \in W$

you should distinguish between the many uses of the symbol 0 according to the context.

identity. 恒等 $\quad I \in \mathcal{L}(V, V)$ $\quad I: V \to V$

$Iv = v$

$I(u+v) = u+v = Iu + Iv$ for all $u, v \in V$.

$I(\lambda v) = \lambda v = \lambda I v$ for all $v \in V$, $\lambda \in \mathbb{F}$

differentiation $\quad D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$

set of polynomials over $\mathbb{R}$

注意：这本书里多项式 $x \in \mathbb{R}$.

$a_n x^n + \cdots + a_1 x + a_0$

$Dp = p'$

$(f+g)' = f' + g'$, $(\lambda f)' = \lambda f'$ (Calculus).

integration $\quad T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$

$Tp = \int_0^1 p(x)\,dx$ $\quad$ (calculus：积分的线性性质)

linear functional 线性泛函 $\quad$ $u\int_a^b f + v\int_a^b g$

linear functional 线性泛函

$$\mu\int_a^b f + v\int_a^b g = \int_a^b \mu f + vg.$$

Thm 3.5 Suppose $v_1, \dots, v_n$ is a basis of V, $w_1, \dots, w_n \in W$.

3.5

Then there exists a unique linear map $T: V \to W$ s.t.

$$Tv_j = w_j \quad \text{for each } j = 1, \dots, n.$$

basis "固定" 一个 map

Pf. Existence

Let $v \in V$

$\Downarrow$ $\exists\, c_1, \dots, c_n \in \mathbb{F}$

$v = c_1 v_1 + \cdots + c_n v_n$

$\Downarrow$

$Tv = T(c_1 v_1 + \cdots + c_n v_n) = c_1 T v_1 + \cdots + c_n T v_n$

Constructive

define T by $T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$

① well-defined

Let $u = v \in V$, then $u = c_1 v_1 + \cdots + c_n v_n$, hence $Tu = Tv$

② Linearity

Let $u, v \in V$

$u = b_1 v_1 + \cdots + b_n v_n$

$v = c_1 v_1 + \cdots + c_n v_n$

$$\begin{aligned}
T(u+v) &= T((b_1+c_1)v_1 + \cdots + (b_n+c_n)v_n) \\
&= (b_1+c_1)w_1 + \cdots + (b_n+c_n)w_n \\
&= (b_1 w_1 + \cdots + b_n w_n) + (c_1 w_1 + \cdots + c_n w_n) \\
&= b_1 T v_1 + \cdots + b_n T v_n + c_1 T v_1 + \cdots + c_n T v_n \\
&= T(b_1 v_1 + \cdots + b_n v_n) + T(c_1 v_1 + \cdots + c_n v_n) \\
&= Tu + Tv
\end{aligned}$$

$$T(\lambda v) = T(\lambda c_1 v_1 + \cdots + \lambda c_n v_n) = \lambda T v$$

Thus T is a linear map from V to W

Uniqueness

Suppose $S \in \mathcal{L}(V, W)$ and $S v_j = w_j$ for $j = 1, \dots, n$

Let $v \in V$, $v = c_1 v_1 + \cdots + c_n v_n$ for some $c_i \in \mathbb{F}$.

Let $v \in V$, $v = c_1 v_1 + \cdots + c_n v_n$ for some $c_i \in \mathbb{F}$.

Then $S(c_1 v_1 + \cdots + c_n v_n) = \underline{c_1 w_1 + \cdots + c_n w_n} = Tv$

now we have $Sv = Tv$ for all $v \in V$

Therefore $S = T$. $\square$

$\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ ☆

Let $A_{jk} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k \in 1, \ldots, n$

$T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$

$$T(x_1, \ldots, x_n) = (A_{11}x_1 + \cdots + A_{1n}x_n, \ \ldots, \ A_{m1}x_1 + \cdots + A_{mn}x_n)$$

Every linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$ is of this form.

$$\begin{pmatrix} A_{11} & A_{12} & A_{1n} \\ \vdots & & \\ A_{m1} & A_{m2} & A_{mn} \end{pmatrix}$$

3 Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

[*The exercise above shows that T has the form promised in the last item of Example 3.4.*]

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}$$

重要！矩阵的定义中会用到.

Pf. choose a basis $e_1, \ldots, e_n$ of $\mathbb{F}^n = \mathbb{R}^n$. $\mathbb{F} = \mathbb{R}$.

$e_1 = (1, 0, \ldots, 0)$

$\cdots$

$e_n = (0, \ldots, 0, 1)$

Let $v = (x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n \in V$.

回想 Thm 3.5 $Te_j = w_j$

目标：找 w_j（合理猜测）.

在 $\mathbb{F}^m$ 中取 n 个向量

$w_1 = (A_{11}, \ldots, A_{m1})$ $\quad A_{jk} \in \mathbb{F}$.

$\cdots$

$w_n = (A_{1n}, \ldots, A_{mn})$

由 Thm 3.5 存在唯一的 $T \in \mathcal{L}(V, W)$ s.t.

$T(x_1 e_1 + \cdots + x_n e_n) = x_1 w_1 + \cdots + x_n w_n$

即 $Te_j = w_j$ $\square$