思考 Let $T: X \to Y$, $S: X \to Y$ Lin map.

$Tx + Sx$	从运算的角度	$Y \times Y$
和		
$(T+S)(x)$	$T+S$	$\mathcal{L}(X,Y) \times \mathcal{L}(X,Y)$
的区别？		

Def $\mathcal{L}(V,W)$ 上的加法，乘法.

Suppose $S, T \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{F}$

sum $S+T$ $\quad (S+T)(v) = Sv + Tv$

product λT $\quad (\lambda T)(v) = \lambda(Tv)$

verify. 我一步也不跳

$S+T$ is linear:

Let $u, v \in V$. (Then $u+v \in V$)

$$(S+T)(u+v) = S(u+v) + T(u+v) \quad \text{by def (sum)}$$
$$= Su + Tu + Sv + Tv \quad \text{Linearity of } T, S$$
$$= (S+T)(u) + (S+T)(v) \quad \text{by def (sum)}$$

$$(S+T)(\lambda v) = S(\lambda v) + T(\lambda v)$$
$$= \lambda(Sv) + \lambda(Tv) \quad \text{Linearity of } T, S$$
$$= \lambda(Sv + Tv) \quad \text{distributivity}$$
$$= \lambda(S+T)(v) \quad \text{by def (sum)} \qquad \square$$

λT is linear

Similarly $\square$

a useful corollary.

$$T(0) = T(0+0) = T(0) + T(0)$$
$$\Rightarrow T(0) = 0$$

Thm 3.7 With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Pf.

addition

commutativity

$$
\begin{aligned}
(S+T)v &= \underset{\overset{\cap}{W}}{Sv} + \underset{\overset{\cap}{W}}{Tv} && \text{sum of maps} \\
&= Tv + Sv && W \text{ is a linear space} \Rightarrow \text{commutative} \\
&= (T+S)v
\end{aligned}
$$

associativity

$$
\begin{aligned}
((S+T)+U)v &= (S+T)v + Uv \\
&= Sv + Tv + Uv \\
&= Sv + (T+U)v \\
&= (S+(T+U))v
\end{aligned}
$$

additive identity

$$
\begin{aligned}
(0+T)v &= 0v + Tv \\
&= Tv = (T+0)v
\end{aligned}
$$

additive inverse

$$
\begin{aligned}
(T+(-T))v &= Tv + (-T)v \\
&= Tv + T(-v) \\
&= T(v-v) \\
&= T0 = 0
\end{aligned}
$$

Scalar multiplication

associativity

$$
\begin{aligned}
(\lambda\mu)Tv &= T(\lambda\mu v) \\
&= \lambda T(\mu v) \\
&= \lambda(\mu Tv)
\end{aligned}
$$

multiplicative identity

$$
\begin{aligned}
\underset{\overset{\cap}{\mathbb{F}}}{1} \cdot Tv &= T(1 \cdot v) \\
&= Tv
\end{aligned}
$$

distributivity

$$
\begin{aligned}
\lambda(S+T)v &= \lambda(Sv + Tv) \\
&= \lambda Sv + \lambda Tv
\end{aligned}
$$

distributivity $\lambda(S+T)v = \lambda(Sv+Tv)$

$$= \lambda Sv + \lambda Tv$$
$$= (\lambda S + \lambda T)v$$

$$(\lambda+\mu)Tv = \lambda Tv + \mu Tv$$
$$= (\lambda T + \mu T)v \qquad \square$$

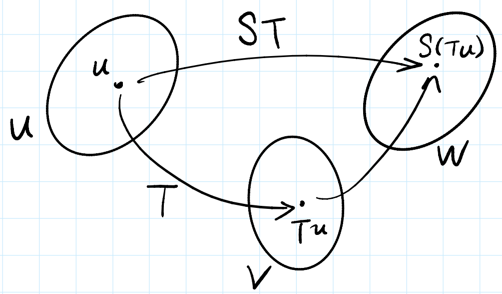
3.8 **Def** Product of Linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$

then the product $ST \in \mathcal{L}(U, W)$

is defined by $(ST)(u) = S(Tu)$

for all $u \in U$.



ST is a linear map from U to W.

pf. Let $u_1, u_2 \in U$, $u \in U$.

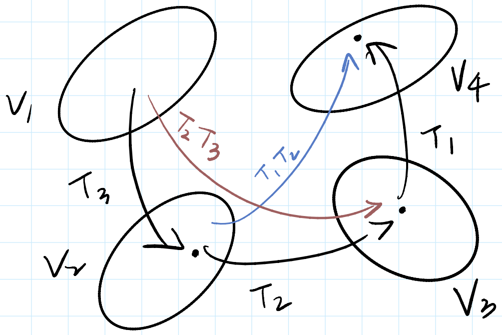
$$(ST)(u_1+u_2) = S(T(u_1+u_2))$$
$$= S(Tu_1 + Tu_2) \quad T \text{ is linear}$$
$$= S(Tu_1) + S(Tu_2) \quad S \text{ is linear}, Tu_i\text{是}V\text{中的向量}$$
$$= (ST)u_1 + (ST)u_2 \quad \text{def of product of maps}$$

$$(ST)(\lambda u) = S(T(\lambda u)) \quad \text{def of prod of maps}$$
$$= S(\lambda(Tu)) \quad T \text{ is linear}$$
$$= \lambda S(Tu) \quad S \text{ is linear}, Tu \in V$$
$$= \lambda(ST)u \quad \text{def of prod of maps.}$$

Algebraic Properties

- associativity

$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$



- identity

$$\underset{\mathcal{L}(V,V)}{T I} = \underset{\mathcal{L}(W,W)}{I T} = T \quad , \quad T \in \mathcal{L}(V,W)$$

- distributivity

$$(S_1 + S_2) T = S_1 T + S_2 T \ ,$$

$$S (T_1 + T_2) = S T_1 + S T_2$$

whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$

$S, S_1, S_2 \in \mathcal{L}(V, W)$

(拓展) Structure : ring

Is it necessary that $ST = TS$?

e.g. $D \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ diff map.

$T \in \mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$

$Tp = x^2 p$.

$$((TD)p)(x) = x^2 p'(x)$$

$$((DT)p)(x) = (x^2 p(x))' = 2x p(x) + x^2 p'(x)$$

$$\Rightarrow \quad TD \neq DT \qquad \square$$

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