HW 2
2020年9月17日 16:34

by MilK
contact QQ812739486 for corrections, many thanks!

1 (1) 任取X中的收敛序列 $\{y_n\}_{n\in\mathbb{N}}$

s.t. $\{y_n\}_{n\in\mathbb{N}}\subset Y$ 且 $\lim\limits_{n\to\infty} y_n = x \in X$.

则 $\{y_n\}_{n\in\mathbb{N}}$ 是Y中的Cauchy列，

由Y完备，$y_n\to y\in Y$ $(n\to\infty)$

由极限唯一性 $x=y$，从而Y中任一收敛序列都收敛于Y中的点，故 Y is closed in X.

(2) Let Y be closed in X,

则Y中任一收敛序列收敛于Y中的点，

又Y中的收敛序列必为Y中的Cauchy列，

Hence Y is complete.

2. Proof by contrapositive

① 若(X,d)无界，

取 $a\in X$，构造$\{x_n\}_{n\in\mathbb{N}}$

s.t. $d(x_n,a)>n \quad \forall n\in\mathbb{N}$.

则 $d(x_{n_k},a)>n_k$，故$\{x_n\}_{n\in\mathbb{N}}$无收敛子列

从而 X is not compact.

② 若(X,d)不完备

则存在X中的Cauchy列 $\{x_n\}_{n\in\mathbb{N}}$

s.t. $x_n\to y\notin X \quad (n\to\infty)$

则任一$\{x_n\}_{n\in\mathbb{N}}$的子列都收敛于 $y\notin X$,

故 X is not compact.

3. 先引入一个定义

E is called totally bounded if, for every $\varepsilon>0$,

E can be covered by finitely many balls of

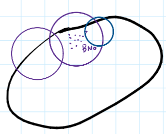
radius ε. (Folland, Real Analysis, Page 15)

我们先证 a 和 b 等价

a. E is complete and totally bounded

b. Every sequence in E has a subsequence that converges to a point of E

(a) => (b): Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of E.



E is totally bounded => E can be covered by finitely many balls of radius 2^{-1},

We denote this collection of balls by $\bigcup\limits_{i=1}^{n} B_i$ for some $n\in\mathbb{N}^+$,

then $E\subset \bigcup\limits_{i=1}^{n} B_i$.

We assert that $\exists N_1\ (1\le N_1\le n)$: B_{N_1} contains infinitely many points of $\{x_n\}_{n\in\mathbb{N}}$,

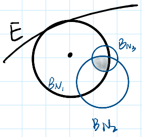
$i=1$

We assert that $\exists N_1\ (1 \le N_1 \le n)$: B_{N_1} contains infinitely many points of $\{x_n\}_{n\in\mathbb{N}}$,

for if every ball contains only finitely many elements of $\{x_n\}_{n\in\mathbb{N}}$ then the sequence is finite.

denote $\{x_n\}_{n\in\mathbb{N}} \cap B_{N_1}$ by $\{x_n\}_{n\in K_1}$, $K_1 \subset \mathbb{N}$.

Similarly, $E \cap B_{N_1}$ can be covered by finitely many balls of radius 2^{-2}, and at least one of these balls contains x_n for infinitely many $n \in K_1$: $x_n \in B_{N_2}$ for $n \in K_2$.



We observe that $\forall x \in B_{N_1}\ \forall y \in B_{N_2}$: $d(x,y) < 2^{-1}+2^{-2}$ since B_{N_2} must intersect B_{N_1}.

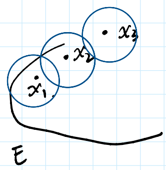
Continuing inductively, we obtain a sequence of balls B_{N_j} with radius 2^{-j}.

Now we pick $n_1 \in K_1$, $n_2 \in K_2$, s.t. $n_1 < n_2 < \cdots$.

Since $d(x_{n_j}, x_{n_{j+p}}) < 2^{-j}+2^{-(j+1)}+\cdots+2^{-(j+p)} < 2^{1-j}$, $p \in \mathbb{N}^+$.

$\{x_{n_j}\}$ is a Cauchy sequence. it converges to a point of E since E is complete

(b) $\Rightarrow$ (a) if E is not totally bounded, then $\exists\, \varepsilon$ s.t. E cannot be covered by finitely many balls of radius ε.



Begin with any $x_1 \in E$, then choose $x_2 \in E \setminus B(x_1;\varepsilon)$,

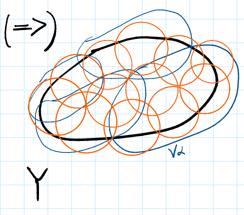
$x_3 \in E \setminus \bigcup_{n=1}^{2} B(x_n;\varepsilon)$, ..., $x_{n+1} \in E \setminus \bigcup_{i}^{n} B(x_j;\varepsilon)$,

then $d(x_n, x_m) > \varepsilon\ \ \forall n, m$, so $\{x_n\}$ has no convergent subsequence

记 C. Y has the finite covering property.

Now we prove that (a) $\Rightarrow$ (c). (就是这道题).

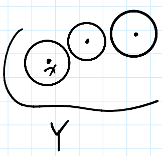
($\Rightarrow$)



Suppose that there exists an open covering $\{V_\alpha\}$ of Y s.t. one cannot extract a finite subcover. But Y can be covered by finitely many balls of radius ε for any $\varepsilon > 0$. Hence there exits an ε-ball that cannot be covered by finitely many open sets in $\{V_\alpha\}$, which is impossible since any open set must contain an ε-ball for some ε.

(c) $\Rightarrow$ (b) (Note that (b) $\Leftrightarrow$ (a))

If $\{x_n\}$ is a sequence in Y with no convergent subsequence,



for each $x \in Y$ there is a ball B_x centered at x that contains x_n for finitely many n (otherwise some subsequence would converge to x). Then $\{B_x\}_{x\in E}$ is an open cover of Y with no finite subcover. □

4. 3的特例.