1.B *Definition of Vector Space*

Properties of addition and scalar multiplication in $\mathbb{F}^n$

(1A. ex12 ~ 16)

12. Show that $(x+y)+z = x+(y+z)$ $\forall x, y, z \in \mathbb{F}^n$

Pf. $x = (x_1, \dots, x_n)$
$y = (y_1, \dots, y_n)$
$z = (z_1, \dots, z_n)$

$$\begin{aligned}(x+y)+z &= (x_1+y_1, \dots, x_n+y_n) + (z_1, \dots z_n) \\ &= (x_1+y_1+z_1, \dots, x_n+y_n+z_n) \qquad x_i+y_i+z_i \in \mathbb{F} \\ &= (x_1+(y_1+z_1), \dots, x_n+(y_n+z_n)) \\ &= (x_1, \dots, x_n) + ((y_1, \dots, y_n)+(z_1, \dots, z_n)) \\ &= x+(y+z) \quad \square.\end{aligned}$$

13. $(ab)x = a(bx)$ $\forall x \in \mathbb{F}^n$, $\forall a, b \in \mathbb{F}$

Pf. $(ab)(x_1, \dots, x_n) = ((ab)x_1, \dots, (ab)x_n)$
$a \in \mathbb{F}, b \in \mathbb{F}, x_i \in \mathbb{F}$.
$$\begin{aligned} &= (a(bx_1), \dots, a(bx_n)) \\ &= a(bx_1, \dots, bx_n) \\ &= a(bx)\end{aligned}$$

$\left.\begin{matrix}12\\13\end{matrix}\right\}$ associativity.

14. $1x = x$ $\forall x \in \mathbb{F}^n$
$1 \in \mathbb{F}$. $x \in \mathbb{F}^n$

Pf. $1(x_1, \dots, x_n) = (1x_1, \dots, 1x_n)$ $\quad 1x_i = x_i$ (multiplicative identity of $\mathbb{F}$)
$$\begin{aligned} &= (x_1, \dots, x_n) \\ &= x. \quad \square\end{aligned}$$

15. $\lambda(x+y) = \lambda x + \lambda y$ $\forall \lambda \in \mathbb{F}$ $\forall x, y \in \mathbb{F}^n$

Pf. $\lambda(x_1+y_1, \dots, x_n+y_n)$
$$\begin{aligned} &= (\lambda(x_1+y_1), \dots, \lambda(x_n+y_n)) \\ &= (\lambda x_1+\lambda y_1, \dots, \lambda x_n+\lambda y_n) \\ &= (\lambda x_1, \dots, \lambda x_n) + (\lambda y_1 + \lambda y_n) \\ &= \lambda(x_1, \dots, x_n) + \lambda(y_1, \dots y_n) \\ &= \lambda x + \lambda y. \quad \square\end{aligned}$$

16. $(a+b)x = ax+bx$ $\forall a, b \in \mathbb{F}$, $\forall x \in \mathbb{F}^n$

16. $(a+b)x = ax+bx \quad \forall a, b \in \mathbb{F}, \forall x \in \mathbb{F}^n$

Pf. $(a+b)(x_1, \dots, x_n)$

$= ((a+b)x_1, \dots, (a+b)x_n)$

$= (ax_1 + bx_1, \dots, ax_n + bx_n)$

$= (ax_1, \dots, ax_n) + (bx_1, \dots, bx_n)$

$= ax + bx. \quad \square$

14, 15 } distributive property.

Commutativity 1.13

additive inverse 1.16

1.18 Def

An addition on V is a function : $(u, v) \mapsto u+v$
$\forall u, v \in V$

A scalar multiplication on V is a function : $(\lambda, v) \mapsto \lambda v$
$\forall \lambda \in \mathbb{F}, \forall v \in V$

1.19 Def.

A vector space is a set V along with

an addition on V and a scalar multiplication on V

satisfying the following properties :

Commutativity

$u+v = v+u \quad \forall u, v \in V$

associativity

$(u+v)+w = u+(v+w)$

$(ab)v = a(bv) \quad \forall u, v, w \in V, \forall a, b \in \mathbb{F}$

additive identity

There exists an element $0 \in V$ s.t. $v+0 = v \quad \forall v \in V$

a unique

additive inverse

$\forall v \in V \ \exists w \in V$ s.t. $v+w=0$

$\exists$ a unique w.

multiplicative identity

$1 \cdot v = v$ for all $v \in V$.

distributive properties

$a(u+v) = au + av$

distributive properties

$$a(u+v) = au+av$$

$$(a+b)v = av+bv \qquad \forall a, b \in \mathbb{F}, \quad \forall u, v \in V.$$

1.20 Def

Elements of a vector space are called vectors or points

V is a vector space (over $\mathbb{F}$)

e.g., $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

$\mathbb{C}^n$ is a vector space over $\mathbb{C}$

Some elementary properties

1.25 A vector space has a unique additive identity.

Pf. Suppose 0 and $0'$ are both additive identities for some vector space V.

Then

$$0' = 0' + 0 \qquad (0 \text{ 是加法单位元})$$

$$= 0 + 0' \qquad (\text{交换律}).\ (\cdots)$$

$$= 0 \qquad (0' \text{ 是加法单位元})$$

$$\Rightarrow 0' = 0. \qquad \square$$

1.26 Every element in a vector space has a unique additive inverse.

Pf. Let V vector space.

$v \in V$. Let w and w' are additive inverses of v.

$$w = w + 0 = w + (v + w')$$

$$= (w+v) + w'$$

$$= 0 + w'$$

$$= w'. \qquad \square$$

1.27 Notation

Let $v, w \in V$. Then

$-v$ denotes the additive inverse of v;

$w - v$ is defined to be $w + (-v)$

1.29 $\underset{\mathbb{F},\ \mathbb{F}^n}{\overset{\in \mathbb{F}}{0}}\, v = \overset{\in V}{0} \quad \forall v \in V$

Pf. $0v = (0+0)v$

Pf. $0v = (0+0)v$
$= 0v + 0v$
$0v - 0v = 0v + 0v - 0v$
$0 = 0v$

1.30 $\underset{\in V}{a0} = \underset{\in V}{0}$ $\forall a \in \mathbb{F},\ 0 \in V$

Pf. $a0 = a(0+0)$
$= a0 + a0$
$\Rightarrow\ a0 = 0$

1.31 $(-1)v = -v$ $\forall v \in V$

Pf. $v + (-1)v = 1v + (-1)v$
$= (1 + (-1))v$
$= 0v$
$= 0$
$\Rightarrow (-1)v = -v$. □

1.23 $\mathbb{F}^S$. 复习一下映射的概念.

If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.

If $f, g \in \mathbb{F}^S$ $\qquad f(x),\ x \in S$.

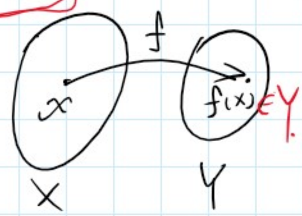
$(f+g)(x) = f(x) + g(x)$ $\forall x \in S$.

$(\lambda f)(x) = \lambda f(x)$. $\forall x \in S$.

$f(x) = x^2$
$\lambda f(x) = \lambda x^2$ $\qquad \mathbb{F}$
$(\lambda f)(x) := \lambda f(x)$ $\qquad \underset{\in}{} \mathbb{F}$.

$(f+g)$
$(f+g)(x) = f(x) + g(x)$.

$f(x) \in \mathbb{F}$.



1.24. $\mathbb{F}^S$ is a vector space.

sketch. $f + g = g + f$. $\quad f: S \to \mathbb{F}$.
$(f+g)(x) = \underbrace{f(x)}_{\in \mathbb{F}} + \underbrace{g(x)}_{\in \mathbb{F}} = g(x) + f(x) = (g+f)(x)$

$(f+g)+h = f+(g+h)$.

$0 \in \mathbb{F}^S$
$0: S \to \mathbb{F}$
$0(x) = \underset{\in}{0}$ $\quad \forall x \in S$.
$(f+0)(x) = f(x) + 0x = f(x)$
$f + 0 = f$.

$\ldots \in \mathbb{F}^S$

$f + 0 = f.$

$(-f) \in \mathbb{F}^S.$

$(-f)(x) = -f(x) \in \mathbb{F}.$

$\in \mathbb{F}.$

$(f + (-f))(x) = f(x) - f(x) = 0.$

$1 \in \mathbb{F}.$

$(1f)(x) = 1f(x). \in \mathbb{F}.$

$= f(x).$

$\lambda(f+g) = \lambda f + \lambda g.$

$(\lambda + \mu) f = \lambda f + \mu f.$