Direct Sums

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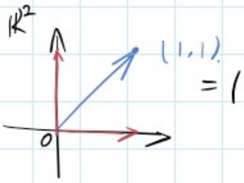
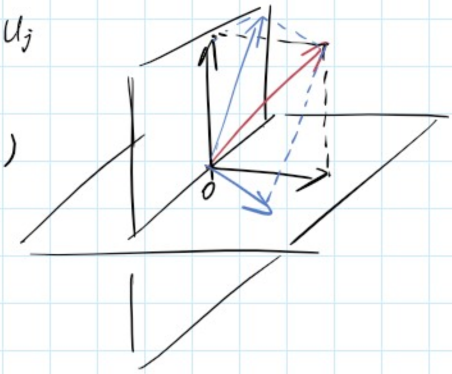
Suppose $U_1, \dots, U_m$ are subspaces of V.
Every element of $U_1 + \cdots + U_m$ can be written in the form

$$u_1 + \cdots + u_m, \quad u_j \in U_j$$



$$= \underset{U}{\underset{\cap}{(1,0)}} + \underset{W}{\underset{\cap}{(0,1)}}$$



$U = \{(x,0): x \in \mathbb{R}\}$
$W = \{(0,y): y \in \mathbb{R}\}$.

1.40 Def

Suppose $U_1, \dots, U_m$ are subspaces of V.

$U_1 + \cdots + U_m$ is called a direct sum if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, each $u_j \in U_j$.

if $U_1 + \cdots + U_m$ is a direct sum, we denote it $U_1 \oplus \cdots \oplus U_m$

1.41 example

Let $U = \{(x,y,0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$
$W = \{(0,0,z) \in \mathbb{F} : z \in \mathbb{F}\}$
Then $\mathbb{F}^3 = U \oplus W$.

Pf. $\mathbb{F}^3 = U + W = \{(x,y,0) + (0,0,z) : x, y, z \in \mathbb{F}\}$
$= \{(x,y,z) : x, y, z \in \mathbb{F}\}$.

$$(x,y,z) = \underset{U}{\underset{\cap}{(x,y,0)}} + \underset{W}{\underset{\cap}{(0,0,z)}}$$

$$(x,y,z) = (x', y', 0) + (0,0,z').$$

$$\mathbb{F}^3 \ni 0 = (x-x', y-y', 0) + (0,0,z-z')$$

$$\underset{\mathbb{F}}{\underset{\cap}{(0,0,0)}} = (\underset{0}{\underset{\|}{x-x'}}, \underset{0}{\underset{\|}{y-y'}}, \underset{0}{\underset{\|}{z-z'}}).$$

$$\Rightarrow x' = x, \ y' = y, \ z' = z. \qquad \square$$

1.42 example

Suppose U_j is the subspace of $\mathbb{F}^n$ s.t.

$U_1 = \{(x, 0, \ldots, 0) : x \in \mathbb{F}\}$

$U_2 = \{(0, x, 0, \ldots, 0) : x \in \mathbb{F}\}$

$\cdots$

Then $\mathbb{F}^n = U_1 \oplus \cdots \oplus U_n$.

pf. $\mathbb{F}^n = U_1 + \cdots + U_n$.

$x \in \mathbb{F}^n$.

$\overset{\shortparallel}{(x_1, x_2, \ldots, x_n)}$.

$x = \underset{\overset{\cap}{U_1}}{(x_1, 0, \ldots, 0)} + \cdots + \underset{\overset{\cap}{U_n}}{(0, \ldots, 0, x_n)}$.

$x = \underset{\overset{\cap}{U_1}}{(x_1', 0, \ldots, 0)} + \cdots + \underset{\overset{\cap}{U_n}}{(0, \ldots, 0, x_n')}$.

$x_i = x_i' \quad i = 1, \ldots, n$. $\square$.

1.43 example

$U_1 \cap U_2 = \{0\}$. $U_1 \cap U_3 = \{0\}$.
$U_2 \cap U_3 = \{0\}$.

Let $U_1 = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$

$U_2 = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$

$U_3 = \{(0, y, y) \in \mathbb{F}^3 : y \in \mathbb{F}\}$

Show that $U_1 + U_2 + U_3$ is not a direct sum.

pf. $\mathbb{F}^3 = U_1 + U_2 + U_3$.

$(0, 0, 0) = \underset{\in U_1}{(0, 1, 0)} + (0, 0, 1) + (0, -1, -1)$

$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$

$U_1 + U_2 + U_3$ is not a direct sum.

1.44 Suppose $U_1, \ldots, U_m$ are subspaces of V.

Then $U_1 + \cdots + U_m$ is a direct sum

$\Leftrightarrow$ the only way to write 0 as a sum

$u_1 + \cdots + u_m$ is by taking each $u_j = 0$.

pf. $(\Rightarrow)$. $U_1 + \cdots + U_m$ is a direct sum.

$\underset{\overset{\cap}{V}}{0} = \underset{\overset{\cap}{U_1}}{0} + \cdots + \underset{\overset{\cap}{U_m}}{0}$ $\Rightarrow$ the only way to write 0 as a sum $U_1 + \cdots + U_m$ is by taking $u_j = 0 \ \forall j$.

$(\Leftarrow)$. $v \in V$.

$v = u_1 + \cdots + u_m, \quad u_j \in U_j$

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$v + (-v) = - - - -$

$$0 = \underset{\substack{\| \\ 0}}{(u_1 - v_1)} + \cdots + \underset{\substack{\| \\ 0}}{(u_m - v_m)}.$$

$\Rightarrow u_i = v_i, \quad i = 1, \ldots, m. \quad \square$.

1.45 Suppose U and W are subspaces of V. Then $U+W$ is a direct sum $\Leftrightarrow U \cap W = \{0\}$.

U_1, U_2, U_3

$U_1 \cap U_2 = \{0\}$

$U_1 \cap U_3 = \{0\}$

$U_2 \cap U_3 = \{0\}$.

$\not\Rightarrow U_1 + U_2 + U_3$ is a direct sum.

pf. $(\Rightarrow)$ $U + W$ is a direct sum.

$v \in U \cap W$

$\Downarrow$

$v \in U$ and $v \in W$.

$\times \Downarrow$

$-v \in U$ and $-v \in W$

$0 = v + (-v) \in U + W$.

$\Downarrow$ $U+W$ is a direct sum.

$v = 0$.

$\Downarrow$

$U \cap W = \{0\}$.

$(\Leftarrow)$ $\underset{\sim}{U \cap W = \{0\}}$.

$0 = \overset{\in U}{u} + w \in W$

$\Downarrow$

$u = -w \in W$.

$u \overset{?}{}$

$\Downarrow$

$u \in U$ and $u \in W$

$u \in U \cap W$.

$\Downarrow$

$u = 0$.

$\Downarrow$

$w = 0. \Rightarrow U + W$ is a direct sum.