Subspaces

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Throughout this book, V denotes a vector space over $\mathbb{F}$.

1.32 Def

A subset U of V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V)

1° $U \subset V$

2° U is also a vector space

1.34 $U \subset V$ is a subspace of V if and only if U satisfies the following conditions

(1) additive identity

$0 \in U$

(2) closed under addition

$u, w \in U \Rightarrow u + w \in U$

(3) closed under scalar multiplication

$(a \in \mathbb{F} \text{ and } u \in U) \Rightarrow au \in U$

V中的加法和数乘.

pf. If U is a subspace of V, then U itself is a vector space.

Conversely, suppose U satisfies the three conditions above.

(1) The additive identity of V is in U

(2) (3)

Recall 1.18

加法和数乘 封闭

$u + w \in U$

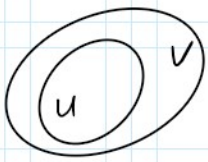
$au \in U$

additive inverse. if $u \in U$, then $-u = (-1)u \in U$ (by (3)).

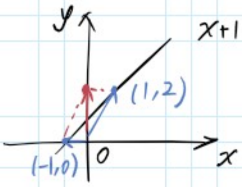
associativity

commutativity

distributivity



Thus U is a vector space. □



$(1,2)+(-1,0)=(0,2)$

$\notin \{(x, x+1) : x \in \mathbb{R}\}$

1.35 example

(a) If $b \in \mathbb{F}$, then

$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$

is a subspace of $\mathbb{F}^4 \iff b = 0$

pf. $(\Rightarrow)$ Let U be a subspace of $\mathbb{F}^4$,

$\Rightarrow \underset{(0,0,0,0)}{\overset{}{0 \overset{}{}}} \in U$, where $0 = (0,0,0,0)$

$\Rightarrow 0 = 0 + b \Rightarrow b = 0$

$(\Leftarrow)$ Let $b = 0$

$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4\}$

(1) $0 \in U$.

(2) $(x_1, x_2, 5x_4, x_4) + (y_1, y_2, 5y_4, y_4)$

$= (x_1 + y_1, x_2 + y_2, 5(x_4 + y_4), x_4 + y_4) \in U$

(3) $\forall a \in \mathbb{F}$, $a(x_1, x_2, 5x_4, x_4)$

$= (ax_1, ax_2, 5ax_4, ax_4) \in U$ $\square$

(b) The set of continuous real-valued functions on $[0,1]$ is a subspace of $\mathbb{R}^{[0,1]}$

$C[0,1]$ is a vector space

pf. (1) 0 (zero mapping)

(2) $f, g \in C[0,1]$

$f + g \in C[0,1]$

(3) $\forall a \in \mathbb{F}$, af

$af \in C[0,1]$. $\square$

$S \rightarrow \mathbb{F}$

$\rightarrow \mathbb{R}^n$

Scalar multiplication on $\mathbb{R}^n$. $\mathbb{R}^n$

Sums of Subspaces

Suppose $U_1, \dots, U_m$ are subspaces of V.

The sum of $U_1, \dots, U_m$

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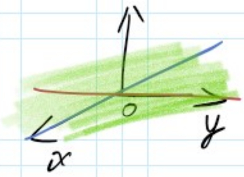
The sum of $U_1, \dots, U_m$

$$U_1 + \cdots + U_m = \{ u_1 + \cdots + u_m : u_1 \in U_1, \dots, u_m \in U_m \}$$

1.37 example



$U = \{ (x, 0, 0) \in \mathbb{F}^3 : x \in \mathbb{F} \}$

$W = \{ (0, y, 0) \in \mathbb{F}^3 : y \in \mathbb{F} \}$

$U + W = \{ (x, y, 0), x, y \in \mathbb{F} \}$.

1.39. Suppose $U_1, \dots, U_m$ are subspaces of V.

Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$

"smallest"

Pf. (1) subspace

$$U_1 + \cdots + U_m = \{ u_1 + \cdots + u_m : u_j \in U_j \}$$

Suppose $w_j \in U_j$, then $w_1 + \cdots + w_m \in U_1 + \cdots + U_m$

$$(u_1 + \cdots + u_m) + (w_1 + \cdots + w_m)$$
$$= (u_1 + w_1) + \cdots + (u_m + w_m) \in U_1 + \cdots + U_m$$

$\Rightarrow$ closed under addition

Similarly, $U_1 + \cdots + U_m$ is closed under scalar multiplication

$0 \in U_j$. $\quad 0 + \cdots + 0 = 0 \in U_1 + \cdots + U_m$.

(2) smallest.

$$(U_1 \cup U_2 \cup \cdots \cup U_m) \subset U_1 + \cdots + U_m$$

Every subspace containing $U_1, \dots, U_m$

denote such a subspace by V.

Then it suffices to show that

$$U_1 + \cdots + U_m \subset V$$
$$\Uparrow$$
$$u_1 + \cdots + u_m \in V$$
$$\Uparrow \ V \text{ is a subspace}$$
$$u_1, \dots, u_m \in V$$

$\square$