Green, Gauss, Stokes.

$$\begin{cases} d\vec{r} = \omega_1 \vec{e_1} + \omega_2 \vec{e_2} \\ d\vec{e_1} = \omega_{12}\vec{e_2} + \omega_{13}\vec{e_3} \\ d\vec{e_2} = \\ d\vec{e_3} = \end{cases}$$

为了揭示基本公式中一次微分形式 ω_i 及 ω_{ij} 之间的关系，我们再次微分（不是简单的二次微分，而是对微分形式进行外微分运算（exterior differentiation））

1）微分形式（自变量 x, y, z）

① 形如 $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$ 的表达式，称为一次微分形式，

简称 1-形式.

微分形式之间定义加法.

设 $\omega = P\,dx + Q\,dy + R\,dz$

$\varphi = f\,dx + g\,dy + h\,dz$

定义 $\omega + \varphi := (f+P)\,dx + (g+Q)\,dy + (h+R)\,dz$

函数 $\lambda = \lambda(x,y,z)$ 与 ω 乘积

$\lambda\omega := \lambda P\,dx + \lambda Q\,dy + \lambda R\,dz$

② 二次微分形式

对于 dx, dy, dz

形式地写出 $dx \wedge dy$

$dx \wedge dz \quad dy \wedge dz \quad dy \wedge dx \quad dz \wedge dy \quad dz \wedge dx$

且规定 $dx \wedge dy = -dy \wedge dx$

$dx \wedge dz = -dz \wedge dx$

$dy \wedge dz = -dz \wedge dy$

且规定 $dx \wedge dx = 0$, $dy \wedge dy = 0$, $dz \wedge dz = 0$ wedge product

称形如 $P(x,y,z)\,dx\wedge dy + Q(x,y,z)\,dx\wedge dz + R(x,y,z)\,dy\wedge dz$

的表达式为二次微分形式，简称为2-形式

类似地，可定义两个2-形式的和 以及 $\lambda(x,y,z)$ 与2-形式 ω 的乘积

③ 三次微分形式

$f(x,y,z)\,dx\wedge dy\wedge dz$ 　　　约定 $dx\wedge dy = -dy\wedge dx$ ...

三次微分形式

2) 微分形式间的外积运算

① 对于 $\varphi_1 = P\,dx + Q\,dy + R\,dz$ 及 $\varphi_2 = f\,dx + g\,dy + h\,dz$

定义一个2-形式记作 $\varphi_1\wedge\varphi_2$

$$(P\,dx + Q\,dy + R\,dz)\wedge(f\,dx + g\,dy + h\,dz)$$

$$= Pg\,dx\wedge dy + Ph\,dx\wedge dz$$
$$+ fQ\,dy\wedge dx + Qh\,dy\wedge dz$$
$$+ fR\,dz\wedge dx + gR\,dz\wedge dy$$
$$= (Pg - fQ)\,dx\wedge dy + (Qh - Rg)\,dy\wedge dz + (Ph - Rf)\,dy\wedge dz$$

类似地，定义1-形式与2-形式外积 及 2-形式与2-形式外积（恒为0）

性质 $(\varphi_1+\varphi_2)\wedge\psi = \varphi_1\wedge\psi + \varphi_2\wedge\psi$

$(f\varphi)\wedge\psi = f\,\varphi\wedge\psi$

若 φ 为 k-form，ψ 为 l-form

$\varphi\wedge\psi = (-1)^{kl}\,\psi\wedge\varphi$

3) 外微分运算

def 对于一个 k-form ω ($k=0,1,2$)，其中约定0次form就是函数 $f(x,y,z)$

定义一个 $k+1$ 次form 记作 $d\omega$

具体规则如下

(1) 若 ω 为0次form，$\omega = f$，定义 $d\omega = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

(2) 若 ω 为 2次form，$\omega = P(x,y,z)\,dx\wedge dy + Q(x,y,z)\,dy\wedge dz + R(x,y,z)\,dz\wedge dx$

定义 $d\omega = dP\wedge dx\wedge dy + dQ\wedge dy\wedge dz + dR\wedge dz\wedge dx$

$$= \frac{\partial P}{\partial z}dz\wedge dx\wedge dy + \frac{\partial Q}{\partial x}dx\wedge dy\wedge dz + \frac{\partial R}{\partial y}dy\wedge dz\wedge dx$$

$$\overset{\text{换序}}{=\!=} \left(\frac{\partial P}{\partial z}+\frac{\partial Q}{\partial x}+\frac{\partial R}{\partial y}\right)dx\wedge dy\wedge dz$$

(3) 1次form $\omega = P\,dx + Q\,dy + R\,dz$

$d\omega := dP\wedge dx + dQ\wedge dy + dR\wedge dz$

外微分算子的基本性质

(i) $d(\varphi+\psi) = d\varphi + d\psi$

(ii) 若 φ 为 k次form 则 $d(\varphi\wedge\psi) = d\varphi\wedge\psi + (-1)^k\varphi\wedge d\psi$

特别地，f 函数 $d(f\psi) = df\wedge\psi + f\,d\psi$

ω 为1次form $d(\omega\wedge\psi) = d\omega\wedge\psi - \omega\wedge d\psi$

(iii) 对任何外形式 ω，

$d(d\omega) = 0$ (零形式)

$d^2 = 0$

Pf. ① $\omega = f(x,y,z)$ (0次form) $d\omega = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$$d(df) = d\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= d\left(\frac{\partial f}{\partial x}dx\right) + d\left(\frac{\partial f}{\partial y}dy\right) + d\left(\frac{\partial f}{\partial z}dz\right)$$

$$= d\left(\frac{\partial f}{\partial x}\right)\wedge dx + d\left(\frac{\partial f}{\partial y}\right)\wedge dy + d\left(\frac{\partial f}{\partial z}\right)\wedge dz$$

$$= \left(f_{xx}\,dx + f_{xy}\,dy + f_{xz}\,dz\right) \wedge dx$$
$$+ \left(f_{yx}\,dx + f_{yy}\,dy + f_{yz}\,dz\right) \wedge dy$$
$$+ \left(f_{zx}\,dx + f_{zy}\,dy + f_{zz}\,dz\right) \wedge dz$$

$$= f_{xy}\,dy\wedge dx + f_{xz}\,dz\wedge dx + f_{yx}\,dx\wedge dy + f_{yz}\,dz\wedge dy + f_{zx}\,dx\wedge dz + f_{zy}\,dy\wedge dz$$
$$= 0$$

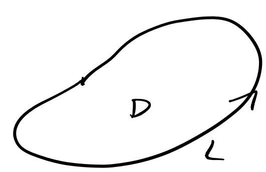
② 若 $w = P\,dx + Q\,dy + R\,dz$

$dw = dP\wedge dx + dQ\wedge dy + dR\wedge dz$ □

利用 differential form 可将3大公式简洁地写出来

① Green's formula

$$\int_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



1-form $w = P(x,y)\,dx + Q(x,y)\,dy$

$$dw = dP\wedge dx + dQ\wedge dy = \frac{\partial P}{\partial y}\,dy\wedge dx + \frac{\partial Q}{\partial x}\,dx\wedge dy$$
$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$$

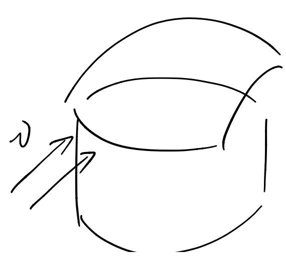
$$\int_{\partial D} w = \iint_D dw$$
(有向区域)

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy = d\sigma\ ?$$
$$\|$$
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\ ?$$

书上有句话：$du\wedge dv$ 可以看成由 du 和 dv 生成的平行四边形的有向面积微元

(定义如此？)

② Gauss 公式



$\vec{v} = P\hat{i} + Q\hat{j} + R\hat{k}$

1- form

$$\iint_{\Sigma} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

$$= \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

$$\omega = P\,dx + Q\,dy + R\,dz$$

$$d\omega = \frac{\partial P}{\partial x} \wedge dx + \frac{\partial Q}{\partial y} \wedge dy + \frac{\partial R}{\partial z} \wedge dz = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

$$\iint_{\partial\Omega} \omega = \iiint_{\Omega} d\omega$$

③ Stokes

...