一、零碎

积分中对二元函数 $z=f(x,y)$，它的全微分长相：$dz = \frac{\partial z}{\partial x}(x,y)\,dx + \frac{\partial z}{\partial y}(x,y)\,dy$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = P\,dx + Q\,dy$

由于 $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ 有 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

若在区域 D 中 $\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y)$

$dz = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$

则 $P\,dx + Q\,dy$ 必为某个 $z=z(x,y)$ 的全微分

一般地，我们不要求 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial x}$，

称 $P\,dx + Q\,dy$ 称为一次微分形式 differential form

二、

在曲面论中，我们希望用 orthonormal basis（切平面）

问题在于曲面上没有一个非常自然的方式选择 orthonormal basis

方案：放弃选取 orthonormal basis "要求"

不做求偏导运算，而用微分运算

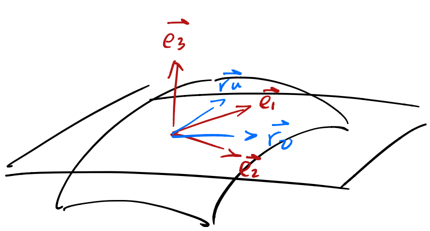
四、曲面上么正标架总是存在的

从自然基出发，用 Schmidt 正交化

取标架 $\{\vec{r}(u,v)\ ;\ \vec{e_1}(u,v), \vec{e_2}(u,v), \vec{e_3}(u,v)\}$

s.t. $\vec{e_3} = \vec{n}$ (unit normal vector)

看 $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ 变化规律（不求偏导，求微分）



$\Sigma:\ \vec{r} = \vec{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))\ ,\ (u,v) \in D$

facts.

(1) 法曲率 $k_n = \frac{\mathrm{II}}{\mathrm{I}}$ ⟹ 特殊的法曲率：主曲率 k_1, k_2 及主方向

$$(EG-F^2)k^2-(GN-2FM+EL)k_n+(LN-M^2)=0$$

主方向 $\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$

Euler 公式 $k_n(\theta) = k_1\cos^2\theta + k_2\sin^2\theta$

(2) 平均曲率 $H=\frac{1}{2}(k_1+k_2)$

Gauss 曲率 $K = k_1k_2$

(3) 曲面上特殊曲线

渐近线（沿着曲线切向，法曲率始终为 0）

$$L\,du^2+2M\,dudv+N\,dv^2=0$$

曲率线　曲线上每点切方向都是该点主方向

方程 $\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$

(4) 基本公式及基本方程

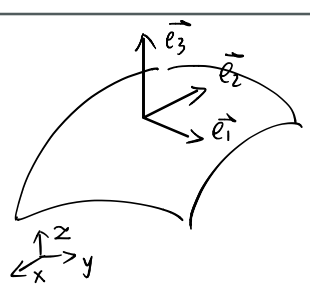
$$\vec{r}_{ij} = \Gamma_{ij}^{k}\,\vec{r}_k + b_{ij}\,\vec{n}$$

$$\vec{n}_i = -b_i^k\,\vec{r}_k$$

$$R_{ijkl} = b_{ik}b_{jl} - b_{il}b_{jk} \qquad \text{Gauss}$$

$$b_{ij,k} = b_{ik,j} \qquad \text{Godazzi}$$

Step. 写出方程，求导运算 ⟶ 得到几何量及几何性质



给定曲面 $\vec{r}=\vec{r}(u,v)=(x(u,v),y(u,v),z(u,v))$

在曲面上选定一个么正标架（orthonormal）$\{\vec{r}(u,v);\ \vec{e_1},\vec{e_2},\vec{e_3}\}$，其中 $\vec{e_3}=\vec{n}$，$\vec{e_i}=\vec{e_i}(u,v)$：$D\to\mathbb{R}^2$

$\vec{e_i}$ depends on u,v；

$\vec{e_i}$ 在 Σ 上"活动"

$\vec{e_i}\cdot\vec{e_j}=\delta_{ij}\quad(i,j=1,2,3)$

首先，么正标架存在（局部）. $\{\vec{r};\ \vec{r_u},\vec{r_v},\vec{n}\}\xrightarrow{\text{Gram-Schmidt}}$ orthonormal

从曲面位置向量 $\vec{r}=\vec{r}(u,v)$ 出发，进行微分运算

设 $\vec{r_u}=a_{11}\vec{e_1}+a_{12}\vec{e_2}$

$\vec{r_v}=a_{21}\vec{e_1}+a_{22}\vec{e_2}$

此处 $a_{ij}=a_{ij}(u,v)$ 为二元函数，设 $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$，nonsingular

$$d\vec{r}=(dx,dy,dz)$$

$x: D\to\mathbb{R}$, $(u,v)\mapsto x(u,v)$ $\quad y: D\to\mathbb{R}$, $(u,v)\mapsto y(u,v)$ $\quad z: D\to\mathbb{R}$, $(u,v)\mapsto z(u,v)$

$$\begin{aligned}&=(x_u du+x_v dv,\ y_u du+y_v dv,\ z_u du+z_v dv)\\&=(x_u,y_u,z_u)du+(x_v,y_v,z_v)dv\\&=\vec{r_u}du+\vec{r_v}dv\end{aligned}$$

$$\begin{aligned}d\vec{r}&=(a_{11}\vec{e_1}+a_{12}\vec{e_2})du+(a_{21}\vec{e_1}+a_{22}\vec{e_2})dv\\&=(a_{11}du+a_{21}dv)\vec{e_1}+(a_{12}du+a_{22}dv)\vec{e_2}\end{aligned}$$

令 $\omega_1=a_{11}du+a_{21}dv,\quad \omega_2=a_{12}du+a_{22}dv$ 一次微分形式

那么有 $d\vec{r}=\omega_1\vec{e_1}+\omega_2\vec{e_2}$

$d\vec{r}=(du)\vec{r_u}+(dv)\vec{r_v}$

事实 $d\vec{r}=\vec{r_u}du+\vec{r_v}dv$，$d\vec{n}=\vec{n_u}du+\vec{n_v}dv$

有 $\mathrm{I}=d\vec{r}\cdot d\vec{r}$，$\mathrm{II}=-d\vec{n}\cdot d\vec{r}$

于是 $\mathrm{I}=(\omega_1\vec{e_1}+\omega_2\vec{e_2})\cdot(\omega_1\vec{e_1}+\omega_2\vec{e_2})$

$=\omega_1\omega_1+\omega_2\omega_2$

现在对 $\vec{e_1},\vec{e_2},\vec{e_3}$ 分别求微分并在 $\{\vec{e_1},\vec{e_2},\vec{e_3}\}$ 下进行分解

有 $d\vec{e_1}=\omega_{11}\vec{e_1}+\omega_{12}\vec{e_2}+\omega_{13}\vec{e_3}$ （微分形式 Combination）

$d\vec{e_2}=\omega_{21}\vec{e_1}+\omega_{22}\vec{e_2}+\omega_{23}\vec{e_3}$ $\quad d\vec{e_i}$ 是微分形式

$$d\vec{e_3} = \omega_{31}\vec{e_1} + \omega_{32}\vec{e_2} + \omega_{33}\vec{e_3}$$

其中 $\omega_{ij} \overset{\text{长相}}{=\!=} p(u,v)\,du + q(u,v)\,dv$ 为一次微分形式

又 $\langle \vec{e_i}, \vec{e_j} \rangle = \delta_{ij}$

$$d(\langle \vec{e_i}, \vec{e_j} \rangle) = 0$$

$$\parallel$$

$$\langle d\vec{e_i}, \vec{e_j} \rangle + \langle \vec{e_i}, d\vec{e_j} \rangle$$

即 $\langle d\vec{e_i}, \vec{e_j} \rangle + \langle \vec{e_i}, d\vec{e_j} \rangle = 0$

$$d\vec{e_i} = \omega_{i1}\vec{e_1} + \omega_{i2}\vec{e_2} + \omega_{i3}\vec{e_3}$$

即 $\omega_{ij} + \omega_{ji} = 0$

$\Rightarrow$ (ω_{ij}) 反对称

于是 $\begin{cases} d\vec{e_1} = \omega_{12}\vec{e_2} + \omega_{13}\vec{e_3} \\ d\vec{e_2} = \omega_{21}\vec{e_1} + \omega_{23}\vec{e_3} \\ d\vec{e_3} = \omega_{31}\vec{e_1} + \omega_{32}\vec{e_2} \end{cases}$ ，其中 $\vec{e_3} = \vec{n}$

$\omega_{ij} + \omega_{ji} = 0$

$\mathrm{II} = -d\vec{r} \cdot d\vec{e_3}$

$= -(\omega_1\vec{e_1} + \omega_2\vec{e_2}) \cdot (\omega_{31}\vec{e_1} + \omega_{32}\vec{e_2})$

$= \omega_1\omega_{13} + \omega_2\omega_{23}$ （变了个号）

$\mathrm{II} = \omega_{13}\omega_1 + \omega_{23}\omega_2$

recall $\omega_1 = a_{11}(u,v)\,du + a_{21}(u,v)\,dv$

$\omega_2 = a_{12}(u,v)\,du + a_{22}(u,v)\,dv$

而 $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ nonsingular,

知 ω_1, ω_2 线性无关

对一次 differential form，规定 $p(u,v)\,du + q(u,v)\,dv = 0 \iff p = q = 0$

即 $\lambda\omega_1 + \mu\omega_2 = 0 \iff \lambda = \mu = 0$

从而任一个一次微分形式都可由 ω_1, ω_2 线性表示

可设 $\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2$

$\omega_{23} = h_{21}\omega_1 + h_{22}\omega_2$

则 $\mathrm{II} = \omega_1(h_{11}\omega_1 + h_{12}\omega_2) + \omega_2(h_{21}\omega_1 + h_{22}\omega_2)$

$$\mathrm{II} = \sum_{i,j=1}^{n} h_{ij}\, \omega_i \omega_j$$

记 $B = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\begin{pmatrix} \vec{r_u} \\ \vec{r_v} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \vec{e_1} \\ \vec{e_2} \end{pmatrix}$

$$\mathrm{I} = \omega_1\omega_1 + \omega_2\omega_2 = (\omega_1 \ \ \omega_2)\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = (du \ \ dv)\, A \left(A^T \begin{pmatrix} du \\ dv \end{pmatrix} \right)$$
$$= (du \ \ dv)(AA^T)\begin{pmatrix} du \\ dv \end{pmatrix}$$

及 $(du \ \ dv)\, A = (\omega_1 \ \ \omega_2)$

另一方面 $\mathrm{I} = (du \ \ dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$

$\Rightarrow \quad AA^T = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

现在

$$\mathrm{II} = \omega_1\omega_{13} + \omega_2\omega_{23} = (\omega_1 \ \ \omega_2)\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$
$$= (du \ \ dv)\, A \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} A^T \begin{pmatrix} du \\ dv \end{pmatrix}$$
$$= (du \ \ dv)\, ABA^T \begin{pmatrix} du \\ dv \end{pmatrix}$$

而 $\mathrm{II} = (du \ \ dv)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$

$\Rightarrow \quad ABA^T = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

由此知 B 为对称矩阵：$h_{12} = h_{21}$

对于 n 阶方阵 X, Y : $tr(XY) = tr(YX)$

$X = \begin{pmatrix} x_{11} & x_{1n} \\ x_{n1} & x_{nn} \end{pmatrix}$ $\quad Y = \begin{pmatrix} y_{11} & y_{1n} \\ y_{n1} & y_{nn} \end{pmatrix}$

$XY = \begin{pmatrix} & \\ & \end{pmatrix}$, $YX = \begin{pmatrix} & \\ & \end{pmatrix}$ 只要看 diagonal

命题. 曲面的平均曲率 $= \frac{1}{2}\operatorname{tr} B = \frac{1}{2}(h_{11}+h_{22})$

Gauss 曲率 $= \det B = h_{11}h_{22} - h_{12}^2$

Since 在自然标架 $\{\vec{r}; \vec{r_u}, \vec{r_v}, \vec{n}\}$, Weingarten 矩阵为

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} = W$$

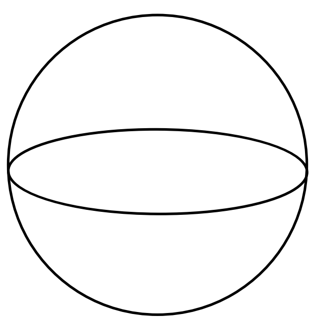
从而 $H = \frac{1}{2}\operatorname{tr} W$, $K = \det W$

现在 $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = AA^T$

$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = ABA^T$

$$\Rightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} = (AA^T)^{-1}(ABA^T) = (A^T)^{-1}A^{-1}ABA^T = (A^T)^{-1}BA^T$$

$$\Rightarrow \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} \sim B$$



$\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2$

$\omega_{23} = h_{12}\omega_1 + h_{22}\omega_2$

$B = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$

☆ 设球心为 $\vec{o}$, 半径 R.

则 $\vec{r}\cdot\vec{r} = R^2$

微分 $d(\vec{r}\cdot\vec{r}) = 0$

$\Rightarrow \langle d\vec{r}, \vec{r}\rangle = 0$

$\Downarrow$ $d\vec{r}$ 为曲面切向

法向与 $\vec{r}$ 平行, 取 $\vec{e_3} = \frac{1}{R}\vec{r}$

$$d\vec{e_3} = d\left(\frac{1}{R}\vec{r}\right) = \frac{1}{R}d\vec{r} = \frac{1}{R}\left(\omega_1\vec{e_1} + \omega_2\vec{e_2}\right)$$

又 $d\vec{e_3} = \omega_{31}\vec{e_1} + \omega_{32}\vec{e_2}$，比较系数，有 $\omega_{31} = \frac{1}{R}\omega_1$， $\omega_{32} = \frac{1}{R}\omega_2$

而 $\underset{\frac{1}{R}\omega_1}{\underset{\parallel}{\omega_{13}}} = h_{11}\omega_1 + h_{12}\omega_2$， $\underset{\frac{1}{R}\omega_2}{\underset{\parallel}{\omega_{23}}} = h_{21}\omega_1 + h_{22}\omega_2$

$\Rightarrow$ $h_{11} = \frac{-1}{R}$， $h_{12} = 0$

$h_{21} = 0$， $h_{22} = \frac{-1}{R}$

$\Rightarrow$ $B = \begin{pmatrix} \frac{-1}{R} & 0 \\ 0 & \frac{-1}{R} \end{pmatrix}$ $\Rightarrow$ $H = \frac{1}{2}\operatorname{tr}B = \frac{-1}{R}$

$K = \det B = \frac{1}{R^2}$

我们已知在自然标架之下，Weingarten 变换的表示矩阵为 $\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$

即 $W(\vec{r_u}) = -\vec{n_u}$

$W(\vec{r_v}) = -\vec{n_v}$

待定系数 $\vec{n_u} = \lambda\vec{r_u} + \mu\vec{r_v}$

$\vec{n_v} = \lambda'\vec{r_u} + \mu'\vec{r_v}$

$$\begin{cases} -L = E\lambda + F\mu \\ -M = F\lambda + G\mu \end{cases} \qquad \begin{pmatrix} -L \\ -M \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L \\ M \end{pmatrix}$$

$$\begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} M \\ N \end{pmatrix}$$

$\vec{n_i} = \lambda_i^k\vec{r_k}$

$\underset{-b_{ij}}{\underset{\parallel}{\vec{n_i}\cdot\vec{r_j}}} = \lambda_i^k g_{kj}$ $\qquad \lambda_i^k g_{kj} = -b_{ij}$

问 $\left(W(\vec{e_1}) \quad W(\vec{e_2})\right) = (\vec{e_1} \quad \vec{e_2})X$， $X = ?$

已知 $\vec{r_u} = a_{11}\vec{e_1} + a_{12}\vec{e_2}$

$\vec{r_v} = a_{21}\vec{e_1} + a_{22}\vec{e_2}$

即 $(\vec{r_u} \quad \vec{r_v}) = (\vec{e_1} \quad \vec{e_2})\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\left(w(\vec{r_u}) \quad w(\vec{r_v})\right) = \left(w(\vec{e_1}) \quad w(\vec{e_2})\right) A^T$$

已知 $\parallel$

$$(\vec{r_u} \quad \vec{r_v})\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} = (\vec{e_1} \quad \vec{e_2}) A^T (AA^T)^{-1}(ABA^T)$$

$$= (\vec{e_1} \quad \vec{e_2}) BA^T$$

$$\Rightarrow \left(w(\vec{e_1}) \quad w(\vec{e_2})\right) A^T = (\vec{e_1} \quad \vec{e_2}) BA^T$$

$$\Rightarrow \left(w(\vec{e_1}) \quad w(\vec{e_2})\right) = (\vec{e_1} \quad \vec{e_2}) B$$

$$\Rightarrow H = \frac{1}{2}\operatorname{tr} B$$

$$K = \det B$$