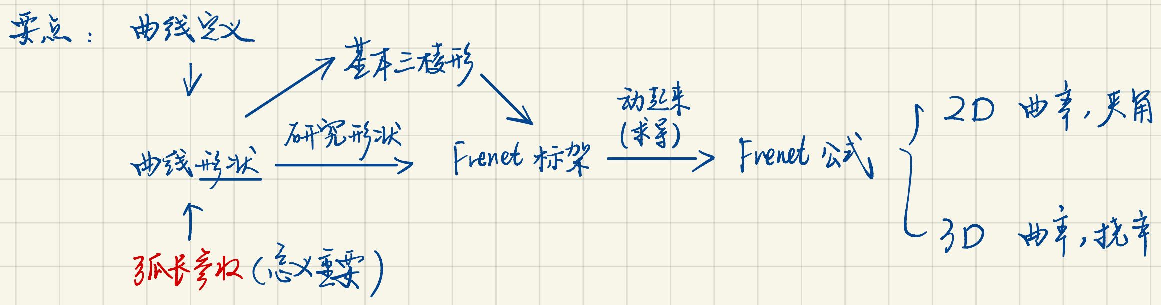
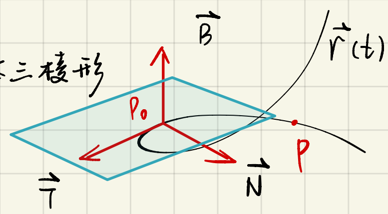
Chap 2. 曲线论.



参考笔记 (可在 personal page 里看).

详细内容 (证明过程见 notes (in personal page))

基本三棱形



切向量 $\vec T$
主法向量 $\vec N$
副法向量 $\vec B$
密切平面

$$\vec T = \vec r'(t_0) = (x'(t_0), y'(t_0), z'(t_0)), \quad \vec r(t_0) = (x_0, y_0, t_0)$$

切线 $$\frac{x-x_0}{x'(t_0)} = \frac{y-y_0}{y'(t_0)} = \frac{z-z_0}{z'(t_0)}$$

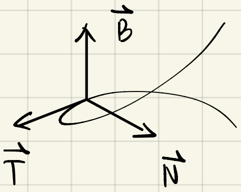
密切平面方程 $\overrightarrow{P_0P}\cdot(\vec r'(t_0)\times\vec r''(t_0)) = 0$，$P$ 为任一点 (x, y, z)

法平面方程 $\overrightarrow{P_0P}\cdot\vec r'(t_0) = 0$

密切平面法向量：$\vec r'(t_0)\times\vec r''(t_0)$

$\Rightarrow$ 从切平面法向量 $\vec r'\times(\vec r'\times\vec r'')$　　外积造正交

由基本三棱形可得 Frenet 标架，只要多一步单位化



$$\vec T = \vec r'(t)$$

$$\vec B = \frac{\vec r'(t)\times\vec r''(t)}{|\vec r'(t)\times\vec r''(t)|}$$ (密切平面的法向量)

$$\vec N = \vec B\times\vec T$$ 注意顺序！

弧长参数 $s(t) = \int_a^t |\vec r'(\tau)|\,d\tau \Rightarrow s'(t) = |\vec r'(t)| > 0$ (正则曲线，其实就是光滑)

$\Rightarrow s(t)\nearrow \Rightarrow$ 有反函数.

弧长参数下的 Frenet frame. $\dot{\vec r}(s)$ 写成 $\dot r(s)$，为了好看.

$$\vec T = \dot r(s), \quad \vec N = \frac{\ddot r(s)}{|\ddot r(s)|}, \quad \vec B = \vec T\times\vec N$$

($\dot r(s)$ 已是单位向量)

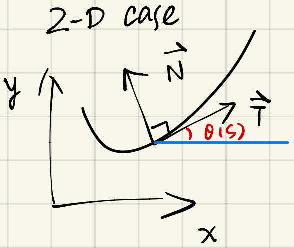
让 Frenet frame 动起来

$\downarrow$ 求导

★ Frenet 公式 (3-dimensional)　　设 $\vec{r}=\vec{r}(s)$，s 弧长参数

$$\begin{cases} \dot{\vec{T}} = \kappa\vec{N} \Rightarrow \ddot{\vec{r}}(s) = \kappa\vec{N} \Rightarrow \kappa = |\ddot{\vec{r}}(s)| \\ \dot{\vec{N}} = -\kappa\vec{T} + \tau\vec{B} \\ \dot{\vec{B}} = -\tau\vec{N} \end{cases}$$

2-D case



$\vec{r} = \vec{r}(s) = (x(s), y(s))$，$s$ 弧长参数

$\dot{\vec{r}}(s) = (\dot{x}(s), \dot{y}(s))$

$\Downarrow$ $\dot{x}(s)^2 + \dot{y}(s)^2 = 1$

写成 $\dot{\vec{r}}(s) = (\cos\theta(s), \sin\theta(s)) := \vec{T}$

$\vec{T}$ 逆时针转 $\frac{\pi}{2}$ 得到 $\vec{N} = \left(\cos(\theta+\frac{\pi}{2}), \sin(\theta+\frac{\pi}{2})\right) = (-\sin\theta(s), \cos\theta(s))$
$= (-\dot{y}, \dot{x})$

not guilty

$\dot{\vec{T}} = \kappa_r\vec{N}$，$\dot{\vec{N}} = -\kappa_r\vec{T}$

$$\dot{\vec{T}} = \frac{d\vec{T}}{ds} = \left(-\sin\theta\frac{d\theta}{ds}, \cos\theta\frac{d\theta}{ds}\right) = \frac{d\theta}{ds}(-\sin\theta, \cos\theta) = \frac{d\theta}{ds}\vec{N}$$

$$\Rightarrow \kappa_r = \frac{d\theta}{ds} \Rightarrow \theta = \int_0^s \kappa_r(u)\,du \Rightarrow \theta(s)$$

$\Downarrow$ $\dot{x} = \cos\theta$, $\dot{y} = \sin\theta$

积出 $\begin{cases} x = x(s) \\ y = y(s) \end{cases}$

$(\ddot{x}, \ddot{y}) = \kappa_r(-\dot{y}, \dot{x})$

$$\begin{cases} \ddot{x} = -\kappa_r\dot{y} \\ \ddot{y} = \kappa_r\dot{x} \end{cases}$$

$\Downarrow$ $\dot{x}^2 + \dot{y}^2 = 1$

$\ddot{y}\dot{x} - \ddot{x}\dot{y} = \kappa_r$

双曲函数

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} = \frac{1-e^{-2x}}{2e^{-x}}.$$

- Hyperbolic cosine: the even part of the exponential function, that is,

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} = \frac{1+e^{-2x}}{2e^{-x}}.$$

Sums of arguments [edit]

$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$
$$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$
$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1+\tanh x\tanh y}$$

particularly

$$\cosh(2x) = \sinh^2 x + \cosh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$$
$$\sinh(2x) = 2\sinh x\cosh x$$
$$\tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$$

Also:

$$\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$
$$\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$$

$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$

没有负号！

$\sinh^2 t - \cosh^2 t = \frac{1}{4}\left[(e^x - e^{-x})^2 - (e^x + e^{-x})^2\right] = -1$

$\cosh^2 t - \sinh^2 t = 1$

5. 证：E^3 正则曲线 $\vec{r}(t)$ 的曲率为 $\kappa(t)=\dfrac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}$ ， $\tau(t)=\dfrac{(\vec{r}',\vec{r}'',\vec{r}''')}{|\vec{r}'\times\vec{r}''|^2}$

一般参数的曲率. Frenet 公式做. (结论要背)

pf. key: Chain Rule

$$\begin{cases}\dot{T}=\kappa\vec{N}\\ \dot{N}=-\kappa\vec{T}+\tau\vec{B}\\ \dot{B}=-\tau\vec{N}\end{cases}$$

弧长 $s(t)=\int_0^t|\vec{r}'(u)|du$

$$\Rightarrow \frac{ds}{dt}=|\vec{r}'(t)| \Rightarrow \frac{dt}{ds}=\frac{1}{|\vec{r}'(t)|}$$

$$\vec{T}(s)=\frac{d\vec{r}}{ds}=\frac{d\vec{r}}{dt}\frac{dt}{ds}=\vec{r}'\frac{dt}{ds},$$

$$\dot{T}(s)=\frac{d}{ds}\left(\vec{r}'\frac{dt}{ds}\right)=\frac{d\vec{r}'}{ds}\frac{dt}{ds}+\vec{r}'\frac{d^2t}{ds^2}$$

$$=\frac{d\vec{r}'}{dt}\frac{dt}{ds}\frac{dt}{ds}+\vec{r}'\frac{d^2t}{ds^2}$$

$$=\vec{r}''\left(\frac{dt}{ds}\right)^2+\vec{r}'\frac{d^2t}{ds^2}$$

$$\dot{T}\times\vec{T}=\kappa\vec{N}\times\vec{T}=-\kappa\vec{B}$$

$|\dot{T}\times\vec{T}|=\kappa$，现在算

$$\dot{T}\times\vec{T}=\left(\vec{r}''\left(\frac{dt}{ds}\right)^2+\vec{r}'\frac{d^2t}{ds^2}\right)\times\left(\vec{r}'\frac{dt}{ds}\right)$$

$$=\vec{r}''\times\vec{r}'\left(\frac{dt}{ds}\right)^3$$

$$=\frac{\vec{r}'(t)\times\vec{r}''(t)}{|\vec{r}'(t)|^3}$$

4. 求曲率 & 挠率　用ex5的公式

(1) $\vec{r}(t) = (a\cosh t,\ a\sinh t,\ bt)$

Sol. $\vec{r}'(t) = (a\sinh t,\ a\cosh t,\ b)$, $\vec{r}''(t) = (a\cosh t,\ a\sinh t,\ 0)$

$$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\sinh t & a\cosh t & b \\ a\cosh t & a\sinh t & 0 \end{vmatrix} = (-ab\sinh t,\ ab\cosh t,\ \underbrace{a^2(\sinh^2 t-\cosh^2 t)}_{=-a^2})$$

$$|\vec{r}'(t)\times\vec{r}''(t)| = (a^2b^2\sinh^2 t + a^2b^2\cosh^2 t + a^4)^{\frac{1}{2}}$$

$$= a(b^2\sinh^2 t + b^2\cosh^2 t + a^2)^{\frac{1}{2}}$$

$$|\vec{r}'(t)| = a(\sinh^2 t + \cosh^2 t)^{\frac{1}{2}}$$

$$|\vec{r}'(t)|^3 = a^3(\sinh^2 t + \cosh^2 t)^{\frac{3}{2}}$$

(2) $\vec{r}(t) = (3t-t^2,\ 3t^2,\ 3t+t^2)$

$\vec{r}'(t) = (3-2t,\ 6t,\ 3+2t)$

$\vec{r}''(t) = (-2,\ 6,\ 2)$

$\vec{r}'''(t) = (0,0,0)$

$$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3-2t & 6t & 3+2t \\ -2 & 6 & 2 \end{vmatrix} = (-18,-12,18)$$

$$|\vec{r}'(t)\times\vec{r}''(t)| = 6\sqrt{22}$$

$$|\vec{r}'(t)|^2 = (3-2t)^2+36t^2+(3+2t)^2$$

$$= 18 + 8t^2+36t^2$$

$$= 44t^2+18 \Rightarrow |\vec{r}'(t)| = (44t^2+18)^{\frac{1}{2}}$$

$$\kappa(t) = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{6\sqrt{22}}{(44t^2+18)^{\frac{3}{2}}}$$

$\tau = 0$. Since 有一行全为0.

13. (1) 求曲率 $\kappa(s)=\dfrac{a}{a^2+s^2}$ （s 是弧长参数）的平面曲线

Sol. $\dfrac{d\theta}{ds}=\dfrac{a}{a^2+s^2}$, $\theta=\displaystyle\int_0^s \frac{a}{a^2+u^2}\,du$

$$=a\int_0^s \frac{1}{a^2\left(1+\left(\frac{u}{a}\right)^2\right)}\,du$$

$$=\int_0^s \frac{1}{1+\left(\frac{u}{a}\right)^2}\,d\left(\frac{u}{a}\right)$$

$$=\arctan\frac{s}{a}$$

$$\dot{r}(s)=(\cos\theta(s),\ \sin\theta(s))=\left(\cos\left(\arctan\frac{s}{a}\right),\ \sin\left(\arctan\frac{s}{a}\right)\right)$$

继续换元：令 $\theta=\arctan\dfrac{s}{a}$, $\tan\theta=\dfrac{s}{a}$, $s=a\tan\theta$, $ds=a\sec^2\theta\,d\theta$

$$\int\cos\left(\arctan\frac{s}{a}\right)ds=\int a\cos\theta\sec^2\theta\,d\theta=a\int\sec\theta\,d\theta=a\int\frac{1}{\cos\theta}\,d\theta$$

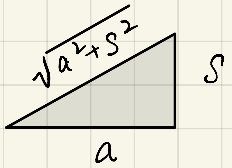
$$=a\log(\sec\theta+\tan\theta)$$

recall $\int\sec\theta\,d\theta$

$$d(\sec\theta+\tan\theta)=(\tan\theta\sec\theta+\sec^2\theta)\,d\theta=\sec\theta(\sec\theta+\tan\theta)\,d\theta$$

$$\int\sec\theta\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta=\int\frac{1}{\sec\theta+\tan\theta}\,d(\sec\theta+\tan\theta)=\log(\sec\theta+\tan\theta)$$

$\tan\theta=\dfrac{s}{a}$



$\sec\theta=\dfrac{\sqrt{a^2+s^2}}{a}$

$$\int\cos\left(\arctan\frac{s}{a}\right)ds=a\log\left(\frac{s+\sqrt{a^2+s^2}}{a}\right)$$

$$\int\sin\left(\arctan\frac{s}{a}\right)ds=\int a\sin\theta\sec^2\theta\,d\theta=a\int\frac{\sin\theta}{\cos^2\theta}\,d\theta=a\int\tan\theta\sec\theta\,d\theta$$

$$=a\sec\theta=\sqrt{a^2+s^2}$$

$$\Rightarrow\ \vec{r}(s)=\left(a\log\left(\frac{s+\sqrt{a^2+s^2}}{a}\right),\ \sqrt{a^2+s^2}\right)$$

(2) $\kappa(s)=\dfrac{1}{\sqrt{a^2-s^2}}$ （s 弧长参数）

解：$\theta(s)=\displaystyle\int_0^s\frac{1}{\sqrt{a^2-s^2}}\,ds=\arcsin\frac{s}{a}$

$\Rightarrow \sin\theta=\dfrac{s}{a}$. $s=a\sin\theta$, $ds=a\cos\theta\,d\theta$

$$\int\sin\theta(s)\,ds=\int\frac{s}{a}\,a\cos\theta\,d\theta=s\sin\theta=s\cdot\frac{s}{a}=\frac{s^2}{a}$$

6. (Not HW) 设: $\vec{r}(s)=\left(\frac{(1+s)^{\frac{3}{2}}}{3}, \frac{(1-s)^{\frac{3}{2}}}{3}, \frac{s}{\sqrt{2}}\right)$ $(-1<s<1)$

以 s 为弧长参数, 并求它的曲率, 挠率和 Frenet 标架

pf. $\frac{d\vec{r}}{ds}=\left(\frac{3}{2}\cdot\frac{(1+s)^{\frac{1}{2}}}{3}, -\frac{3}{2}\cdot\frac{(1-s)^{\frac{1}{2}}}{3}, \frac{1}{\sqrt{2}}\right)$

$\left|\frac{d\vec{r}}{ds}\right|^2=\frac{1}{4}(1+s)+\frac{1}{4}(1-s)+\frac{1}{2}=1 \Rightarrow \vec{r}(s)$ 的参数 s 为弧长参数.

$\dot{r}(s)=\left(\frac{1}{2}(1+s)^{\frac{1}{2}}, -\frac{1}{2}(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)=\vec{T}$

$\ddot{r}(s)=\left(\frac{1}{4}(1+s)^{-\frac{1}{2}}, \frac{1}{4}(1-s)^{-\frac{1}{2}}, 0\right)$

$|\ddot{r}(s)|=\left(\frac{1}{16}(1+s)^{-1}+\frac{1}{16}(1-s)^{-1}\right)^{\frac{1}{2}}=\frac{1}{4}\left((1+s)^{-1}+(1-s)^{-1}\right)^{\frac{1}{2}}=\frac{\sqrt{2}}{4}(1-s^2)^{-\frac{1}{2}}$

$\vec{N}=\frac{\ddot{r}(s)}{|\ddot{r}(s)|}=\frac{\left((1+s)^{-\frac{1}{2}}, (1-s)^{-\frac{1}{2}}, 0\right)}{\left((1+s)^{-1}+(1-s)^{-1}\right)^{\frac{1}{2}}}=\frac{1}{\sqrt{2}}(1+s)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}\left((1+s)^{-\frac{1}{2}}, (1-s)^{-\frac{1}{2}}, 0\right)$

$=\frac{1}{\sqrt{2}}\left((1-s)^{\frac{1}{2}}, (1+s)^{\frac{1}{2}}, 0\right)$, $\dot{N}=\frac{1}{2\sqrt{2}}\left(-(1-s)^{-\frac{1}{2}}, (1+s)^{-\frac{1}{2}}, 0\right)$

$\vec{B}=\vec{T}\times\vec{N}=\frac{-1}{\sqrt{2}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (1-s)^{\frac{1}{2}} & (1+s)^{\frac{1}{2}} & 0 \\ \frac{1}{2}(1+s)^{\frac{1}{2}} & -\frac{1}{2}(1-s)^{\frac{1}{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}=-\left(\frac{1}{2}(1+s)^{\frac{1}{2}}, \frac{1}{2}(1-s)^{\frac{1}{2}}, -\frac{1}{\sqrt{2}}\right)$ $\left(-\frac{1}{2}(1+s)^{\frac{1}{2}}, \frac{1}{2}(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)$

$\dot{B}=\left(-\frac{1}{4}(1+s)^{-\frac{1}{2}}, -\frac{1}{4}(1-s)^{-\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)$

Frenet 公式: $\begin{cases} \dot{T}= \quad \kappa\vec{N} \\ \dot{N}=-\kappa\vec{T}+ \quad +\tau\vec{B} \\ \dot{B}= \quad -\tau\vec{N} \end{cases}$

$\dot{B}\cdot\vec{N}=\frac{1}{4\sqrt{2}}\left(-\frac{\sqrt{1-s}}{\sqrt{1+s}}-\frac{\sqrt{1+s}}{\sqrt{1-s}}\right)$

$=-\frac{1}{4\sqrt{2}}\frac{1-s+1+s}{\sqrt{1-s^2}}=\frac{-1}{2\sqrt{2}}(1-s^2)^{-\frac{1}{2}}$

$\Rightarrow$ 作内积 $\dot{B}\cdot\vec{N}=-\tau$

$\tau=\frac{1}{2\sqrt{2}}(1-s^2)^{-\frac{1}{2}}$

得 $\tau=\frac{1}{2\sqrt{2}}\frac{s}{\sqrt{1-s^2}}$