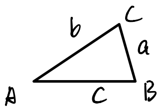
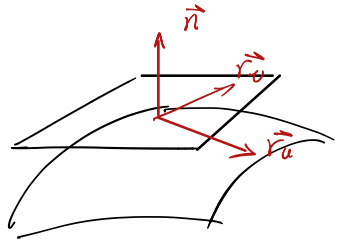
1) Motivation





6个基本量有什么联系？

考察曲面上自然标架 $\Sigma = \{\vec{r};\ \vec{r_u}, \vec{r_v}, \vec{n}\}$ 的变化规律

$\vec{r}_{uu}$, $\vec{r}_{uv}$, $\vec{r}_{vv}$, $\vec{n}_u$, $\vec{n}_v$ 都用 basis $\{\vec{r_u}, \vec{r_v}, \vec{n}\}$ 线性表示

Let $\vec{r}_{uu} = \lambda \vec{r_u} + \mu \vec{r_v} + \nu \vec{n}$

$$L = \vec{n}\cdot\vec{r}_{uu} = 0 + 0 + \nu$$

$$\vec{r}_{uu}\cdot\vec{r_u} = \lambda E + \mu F + 0$$

$$\| $$

$$\frac{1}{2}(\vec{r_u}\cdot\vec{r_u})_u = \frac{1}{2}E_u \Rightarrow \lambda E + \mu F = \frac{1}{2}E_u$$

$$\vec{r}_{uu}\cdot\vec{r_v} = \lambda F + \mu G + 0 \Rightarrow \lambda E + \mu G = F_u - \frac{1}{2}E_v$$

$$(\vec{r_u}\cdot\vec{r_v})_u = \vec{r}_{uu}\cdot\vec{r_v} + \vec{r_u}\cdot\vec{r}_{uv}$$

$$F_u = \vec{r}_{uu}\cdot\vec{r_v} + \frac{1}{2}(\vec{r_u}\cdot\vec{r_u})_v$$

$$\vec{r}_{uu}\cdot\vec{r_v} = F_u - \frac{1}{2}E_v$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{pmatrix}$$

$$\begin{cases} \vec{r}_{uu} = \dfrac{E_u}{2E}\vec{r_u} - \dfrac{E_v}{2G}\vec{r_v} + L\vec{n} \\[2ex] \vec{r}_{uv} = \dfrac{E_v}{2E}\vec{r_u} + \dfrac{G_u}{2G}\vec{r_v} + M\vec{n} \\[2ex] \vec{r}_{vv} = -\dfrac{G_u}{2E}\vec{r_u} + \dfrac{G_v}{2G}\vec{r_v} + N\vec{n} \end{cases}$$

利用 $\vec{r}_{uuv} = \vec{r}_{uvu}$，$\vec{r}_{vvu} = \vec{r}_{vuv}$ 可得 E, F, G, L, M, N 的关系

Weingarten: $(\vec{n_u}\ \ \vec{n_v}) = (\vec{r_u}\ \ \vec{r_v})\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$

2) 张量记号初步

对曲面 Σ : $\vec{r}=\vec{r}(u,v)$, $(u,v)\in D$

认为 u 是第一个参数，记作 u^1,

v 是第二个参数，记作 u^2

相应地，偏导记号: $\vec{r_u} := \vec{r_1}$, $\vec{r_v} := \vec{r_2}$; $\vec{r_1}=\dfrac{\partial\vec{r}}{\partial u^1}(u^1,u^2)$, $\vec{r_2}=\dfrac{\partial\vec{r}}{\partial u^2}(u^1,u^2)$

$\vec{r}_{uuv} := \vec{r}_{112}$

克氏记号 (Christoffel Symbol)

第一类克氏记号

the first fundamental form

$$E=\vec{r_u}\cdot\vec{r_u}=\vec{r_1}\cdot\vec{r_1}:=g_{11}$$

$$F=\vec{r_u}\cdot\vec{r_v}=\vec{r_1}\cdot\vec{r_2}:=g_{12}=g_{21}$$

$$G=\vec{r_v}\cdot\vec{r_v}=\vec{r_2}\cdot\vec{r_2}:=g_{22}$$

于是 $\begin{pmatrix} E & F \\ F & G \end{pmatrix}=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$

对于指标 i,j,k (取值于 $\{1,2\}$)

定义 $[ij,k]=\dfrac{1}{2}\left(\dfrac{\partial g_{ik}}{\partial u^j}+\dfrac{\partial g_{jk}}{\partial u^i}-\dfrac{\partial g_{ij}}{\partial u^k}\right)$ 这个要背!

称 $[ij,k]$ 为第一类克氏记号 (the first Christoffel Symbol)

对于 $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}=\begin{pmatrix} E & F \\ F & G \end{pmatrix}$

它的 inverse 记为 $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}:=\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} g_{11}g^{11}+g_{12}g^{21} & g_{11}g^{12}+g_{12}g^{22} \\ g_{21}g^{11}+g_{22}g^{21} & g_{21}g^{12}+g_{22}g^{22} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \sum_{k=1}^{2} g_{ik}g^{kj} = \delta_i^j, \quad \sum_{s=1}^{2} g^{is}g_{sj} = \delta_j^i$$ （干嘛要写2个？）

直接写为 $g_{ik}g^{kj} = \delta_i^j$ 及 $g^{is}g_{sj} = \delta_j^i$ （此为 Einstein 求和约定）

上下标"碰"，省略 Σ.

第二类克氏记号

记 $\Gamma_{jk}^{i} = g^{il}[jk,l] = g^{i1}[jk,1] + g^{i2}[jk,2]$

上下指标碰 —> 求和

代几个 ijk 进去看看：

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$ ，记 $g = \det(g^{\alpha\beta})$

$$\Gamma_{11}^{2} = g^{21}[11,1] + g^{22}[11,2]$$

$$= g^{21}\frac{1}{2}\left(\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1}\right) + g^{22}\frac{1}{2}\left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right)$$

$$= g^{22}\frac{\partial g_{12}}{\partial u^1} + \frac{1}{2}\left(g^{21}\frac{\partial g_{11}}{\partial u^1} - g^{22}\frac{\partial g_{11}}{\partial u^2}\right)$$

$$= \frac{E}{g}\frac{\partial F}{\partial u} + \frac{1}{2}\left(\frac{-F}{g}\frac{\partial E}{\partial u} - \frac{E}{g}\frac{\partial E}{\partial v}\right)$$

记号方面，求解 Linear System

$$\begin{cases} b^{11}\cdot(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) = c_1 b^{11} \\ b^{12}\cdot(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) = c_2 b^{12} \\ \vdots \\ b^{1n}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) = c_n b^{1n} \end{cases}$$

原方程组写成 $Ax = c$

记 $A = (a_{ij})$，将 A^{-1} 记作 (b^{ij}) 则有 $\sum_{k=1}^{n} a_{ik} b^{kj} = \delta_i^j$ 利用 $(a_{ij})(b^{ij}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

先看一个简单情形： $a_{ik} b^{kj} = \delta_i^j$

相加，$(b^{11}a_{11} + b^{12}a_{21} + \cdots + b^{1n}a_{n1})\, x_1$

$+ (b^{11}a_{12} + b^{12}a_{22} + \cdots + b^{1n}a_{n2})\, x_2$

$+ \cdots + (b^{11}a_{1n} + b^{12}a_{2n} + \cdots + b^{1n}a_{nn})\, x_n = \sum_{k=1}^{n} b^{1k} c_k$

即 $(b^{1k}a_{k1})\, x_1 + \cdots + (b^{1k}a_{kn})\, x_n = b^{1k} c_k$

$$\Rightarrow \quad x_1 = b^{1k} c_k$$

$$x_2 = b^{2k} c_k$$

$$\vdots$$

$$x_n = b^{nk} c_k$$

(Appendix)

Einstein 求和约定

在一个单项式中，若一个指标字母作为上标和下标各出现一次，则该式就表示是对 $\alpha = 1, 2$ 的求和式

$$d\vec{r} = \vec{r}_1 du^1 + \vec{r}_2 du^2 = \vec{r}_\alpha du^\alpha$$

上下指标多对重复出现就表示该式是多重求和式

$$I = g_{11} du^1 du^1 + 2 g_{12} du^1 du^2 + g_{22} du^2 du^2$$

$$= g_{11}du^1du^1 + g_{12}du^1du^2 + g_{21}du^1du^2 + g_{22}du^2du^2$$

$$= g_{1\beta}du^1du^\beta + g_{2\beta}du^2du^\beta$$

$$= g_{\alpha\beta}du^\alpha du^\beta$$

$$\mathrm{II} = b_{\alpha\beta}du^\alpha du^\beta$$

另记 $(g_{\alpha\beta})^{-1} = (g^{\alpha\beta})$, $g = \det(g_{\alpha\beta})$, $b = \det(b_{\alpha\beta})$

矩阵运算 $(g^{\alpha\beta})\cdot(g_{\alpha\beta}) = (\delta^\alpha_\beta)$

分量: $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$