向量分析

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vector calculus

$\mathbb{R} \to \mathbb{R}^3 \qquad \mathbb{R} \to \mathbb{R}^n$

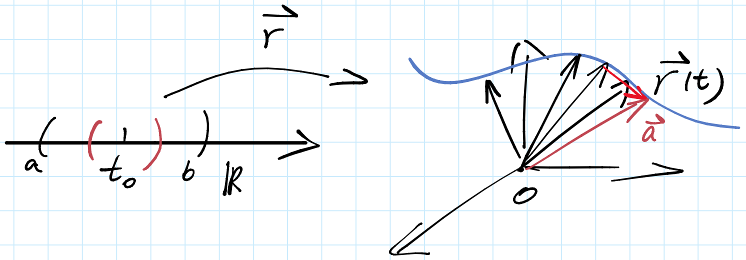
1) vector-valued function

对 $I=(a,b)$

如果有一个对应法则 $\vec{r}:(a,b) \longrightarrow E^3$ （通常的3维空间）

$$t \mapsto \vec{r}(t)$$

$\forall t\in(a,b),\ \vec{r}(t) \in E^3.$



2) limit

let $\vec{r}:(a,b) \longrightarrow E^3$ be a vector valued function

当 $t \to t_0$ 时 $\vec{r}(t)$ 以 $\vec{a}$ 为极限

$\forall \varepsilon>0\ \exists \delta>0\ \forall t\ (0<|t-t_0|<\delta):\ |\vec{r}(t)-\vec{a}| < \varepsilon$

denoted $\lim\limits_{t\to t_0} \vec{r}(t) = \vec{a}$

3) def 对 $\vec{r}=\vec{r}(t)\quad t\in(a,b)$

if for $t_0\in(a,b)$: $\lim\limits_{t\to t_0}\vec{r}(t)=\vec{r}(t_0)$

称 $\vec{r}(t)$ is continuous at t_0.

4) Differentiation

$t_0\in(a,b)$.

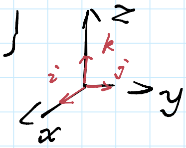
$$\frac{\vec{r}(t_0+\Delta t)-\vec{r}(t_0)}{\Delta t}$$

令 $\Delta t \to 0$. if $\lim\limits_{\Delta t\to 0} \dfrac{\Delta \vec{r}}{\Delta t}$ 存在, then 称其为 $\vec{r}$ 的导数 at t_0,

denoted $\vec{r}\,'(t_0) = \lim\limits_{\Delta t\to 0} \dfrac{1}{\Delta t}\left(\vec{r}(t_0+\Delta t)-\vec{r}(t_0)\right)$

denoted $\vec{r}'(t_0) = \lim\limits_{\Delta t \to 0} \frac{1}{\Delta t}\left(\vec{r}(t_0+\Delta t) - \vec{r}(t_0) \right)$

注：取定直角坐标系 $\Sigma = \{o, x, y, z\}$



那么

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

where $x(t), y(t), z(t)$ 为 $(a,b) \to \mathbb{R}$

$\vec{r}(t) = (x(t), y(t), z(t))$.

Prop 1 对 $\vec{r}(t) = (x(t), y(t), z(t))$ 及 $\vec{a} = (a_1, a_2, a_3)$

$$\lim_{t\to t_0} \vec{r}(t) = \vec{a} \iff \left(\lim_{t\to t_0} x(t) = a_1 \ \wedge \ \lim_{t\to t_0} y(t) = a_2 \ \wedge \ \lim_{t\to t_0} z(t) = a_3 \right)$$

$$\vec{r}(t) - \vec{a} = (x(t)-a_1)\vec{i} + (y(t)-a_2)\vec{j} + (z(t)-a_3)\vec{k}$$

$$|\vec{r}(t) - \vec{a}| = \left[(x(t)-a_1)^2 + (y(t)-a_2)^2 + (z(t)-a_3)^2 \right]^{\frac{1}{2}}$$

$$|x(t)-a_1| \le |\vec{r}(t) - \vec{a}| \to 0$$

$$|y(t)-a_2| \le |\vec{r}(t) - \vec{a}| \to 0$$

$$|z(t)-a_3| \le |\vec{r}(t) - \vec{a}| \to 0$$

Thm 2. Coord-wise continuous

Prop 3. $\vec{r}'(t)$ 可导 at t_0 $\iff$ x, y, z 均可导 at t_0, $\vec{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$.

Algebraic properties

Thm Let $\vec{r}, \vec{r}_1$ 均可导.

① $[\vec{r} + \vec{r}_1]' = \vec{r}' + \vec{r}_1'$

② $[\vec{r} \cdot \vec{r}_1]' = \vec{r}' \cdot \vec{r}_1 + \vec{r} \cdot \vec{r}_1'$

③ $(\vec{r} \times \vec{r}_1)' = \vec{r}' \times \vec{r}_1 + \vec{r} \times \vec{r}_1'$

③ $(\vec{r} \times \vec{r}_1)' = \vec{r}\,' \times \vec{r}_1 + \vec{r} \times \vec{r}_1\,'$

④ $(\vec{a}(t), \vec{b}(t), \vec{c}(t))'$

$$\parallel$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}'$$

⑤ 对 $f:(a,b) \to \mathbb{R}$ 及向量值 $\vec{r}(t)$

$$\Big(\underset{\mathbb{R}}{\underset{\in}{f(t)}}\ \underset{\mathbb{R}^3}{\underset{\in}{\vec{r}(t)}}\Big)' = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}\,'(t)$$

scalar multiplication

<u>Thm</u>. $\vec{r} = \vec{r}(t)$, $t \in I$.

(1) $\vec{r}(t)$ 长度固定 $\Longleftrightarrow$ $\vec{r}\,'(t) \perp \vec{r}(t)$ $\forall t$

Pf. $|\vec{r}(t)| = a$

$$\Updownarrow$$

$$\vec{r}(t) \cdot \vec{r}(t) = a^2$$

$$\Updownarrow$$

$$(\vec{r}(t) \cdot \vec{r}(t))' = 0 \Longleftrightarrow \vec{r}\,' \cdot \vec{r} + \vec{r} \cdot \vec{r}\,' = 0$$