1) 一般地，对于非空集合 X，从 X 到 X 的一个一一映射，称 X 的一个变换

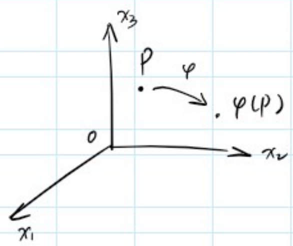
2) 设 E^3 为 3 维 Euclidean space
若 E^3 上一个变换 φ (one-to-one onto)
满足 $d(P,Q) = d(\varphi(P), \varphi(Q))$
则称 φ 是 E^3 中的一个等距变换.

3) isometry in E^3

Thm 取定 E^3 中一个直角坐标系 $\Sigma = \{O, x, y, z\}$
则 E^3 中的一个变换 $\varphi: E^3 \to E^3$

$$P \mapsto \varphi(P)$$
$$(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3)$$

$\varphi: \begin{cases} y_1 = f_1(x_1, x_2, x_3) \\ y_2 = f_2(x_1, x_2, x_3) \\ y_3 = f_3(x_1, x_2, x_3) \end{cases}$ is isometric $\iff$ φ 表达式为
$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3$$



where $A = (a_{ij})$ is a 3×3 orthogonal matrix
$y = Ax + b$, $AA^T = I$

Pf 2. 设 $\varphi: E^3 \to E^3$ isometry
考虑 $F: E^3 \to E^3$
$$P \mapsto \varphi(P) - \varphi(0)$$
则 $F(0) = 0$ 且 F is isometry: $d(P,Q) = |P - Q| = |\varphi(P) - \varphi(Q)|$
$$= |\varphi(P) - \varphi(0) - (\varphi(Q) - \varphi(0))|$$
$$= |F(P) - F(Q)|$$

故考虑 isometry $F: E^3 \to E^3$, $F(0) = 0$
$$F(P) = \varphi(P) - \varphi(0)$$
$$\varphi(P) = F(P) + \varphi(0)$$

设 F 表达式 $y = f(x)$. 首先，注意到对于 E^3 中 3 点 A, B, C,
总有 $d(A,C) \le d(A,B) + d(B,C)$
且 $d(A,C) = d(A,B) + d(B,C) \iff A, B, C$ 共线, B lies between A and C.

$\forall P \in E^3$ 记 $P' = F(P)$
$d(O,P) = d(F(0), F(P)) = d(O, F(P))$
对 $0 < t < 1$, 有 $d(O,P) = d(O,tP) + d(tP, P)$

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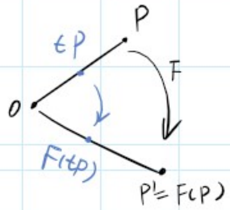
对 $0<t<1$，有 $d(O,P) = d(O,tP) + d(tP, P)$



$$\Downarrow$$

$$d(O, F(P)) = d(O, F(tP)) + d(F(tP), F(P))$$

主要就是搞这个 F(tp)

$$\Downarrow$$

O，$F(tP)$，$F(P)$ 共线，且 $F(tP)$ 介于 O 及 $F(P)$

$$\Downarrow$$

$$F(tP) = k F(P), \quad 0<k<1.$$

we assert that $k = t$, i.e. $F(tP) = t F(P)$

$$d(O,tP) = |tP| = t|P|$$

$$\|$$

$$d(F(O), F(tP)) = d(O, F(tP)) = |F(tP)| = |k F(P)| = k|F(P)| = k|P|$$

isometry

generally, consider homogeneous function

$f(tX) = t f(X)$, $X \in E^3$ here.

$$f(tx_1, tx_2, tx_3) = t f(x_1, x_2, x_3)$$

differentiate over t

$$x_1 f_1'(tx_1, tx_2, tx_3) + x_2 f_2'(\quad) + x_3 f_3'(\quad) = f(x_1, x_2, x_3)$$

Let $t \to 0$

$$x_1 f_1'(0) + x_2 f_2'(0) + x_3 f_3'(0) = f(x_1, x_2, x_3)$$

$$\Downarrow$$

$$f(x_1, x_2, x_3) = \sum_{i=1}^{3} a_{1i} x_i$$

$$\Downarrow$$

$$y = Ax$$

保距 $\Rightarrow$ $y^T y = x^T x$

$$(Ax)^T(Ax) = x^T A^T A x \Rightarrow A^T A = I \qquad \square$$

φ 诱导 $\sigma: E^3 \to E^3$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

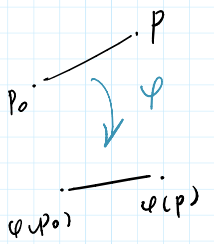
σ is a linear map

Pf 2. 设 isometry φ 的直角坐标表达式为

$$\begin{cases} x' = f(x, y, z) \\ \ldots \end{cases}$$

由于 φ 为 isometric，对于任 2 对对应点

$$\begin{cases} x' = f(x,y,z) \\ y' = g(x,y,z) \\ z' = h(x,y,z) \end{cases}$$



由于 φ 是 isometric，对于任2对 对应点

$p_0(x_0, y_0, z_0)$ ， $\varphi(p_0): (x_0', y_0', z_0')$

$p(x, y, z)$ ， $\varphi(p): (x', y', z')$

有 $|p_0 p| = |\varphi(p_0)\varphi(p)|$

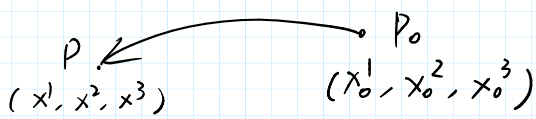
$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = (f(x,y,z)-f_0)^2+(g(x,y,z)-g_0)^2+(h(x,y,z)-h_0)^2$$

设 φ: $$\begin{cases} y^1 = y^1(x^1, x^2, x^3) \\ y^2 = y^2(x^1, x^2, x^3) \\ y^3 = y^3(x^1, x^2, x^3) \end{cases}$$

记 $y^1(x_0^1, x_0^2, x_0^3) = y_0^1$

$y^2(x_0^1, x_0^2, x_0^3) = y_0^2$

$y^3(x_0^1, x_0^2, x_0^3) = y_0^3$



有 $(y^1-y_0^1)^2+(y^2-y_0^2)^2+(y^3-y_0^3)^2 = (x^1-x_0^1)^2+(x^2-x_0^2)^2+(x^3-x_0^3)^2$

$$\sum_{i=1}^{3}\left(y^i(x_1,x_2,x_3)-y_0^i\right)^2 = \sum_{i=1}^{3}(x^i-x_0^i)^2$$

对 x_k 求导

$$\sum_{i=1}^{3} 2(y^i-y_0^i)\cdot\frac{\partial y^i}{\partial x_k} = 2\sum_{i=1}^{3}(x^i-x_0^i)\cdot\frac{\partial}{\partial x^k}(x^i-x_0^i)$$

$$\sum_{i=1}^{3}(y^i-y_0^i)\frac{\partial y^i}{\partial x^k} = \sum_{i=1}^{3}(x^i-x_0^i)\,\delta_k^i = x^k-x_0^k$$

对 x^j 求导.

$$\sum_{i=1}^{3}\frac{\partial y^i}{\partial x^j}\frac{\partial y^i}{\partial x^k} + \sum_{i=1}^{3}(y^i-y_0^i)\frac{\partial^2 y^i}{\partial x^k\partial x^j} = \delta_{jk}$$

在 $P_0(x_0^1, x_0^2, x_0^3)$ 处 $\displaystyle\sum_{i=1}^{3}\frac{\partial y^i}{\partial x^j}(P_0)\frac{\partial y^i}{\partial x^k}(P_0) = \delta_{jk}$

对 x^l 求导

$$\sum_{i=1}^{3}\frac{\partial^2 y^i}{\partial x^j\partial x^l}\frac{\partial y^i}{\partial x^l}+\sum_{i=1}^{3}$$

$$\sum_{i=1}^{3}\frac{\partial^2 y^i}{\partial x^j\partial x^l}\cdot\frac{\partial y^i}{\partial x^k}+\sum_{i=1}^{3}\frac{\partial y^i}{\partial x^j}\frac{\partial^2 y^i}{\partial x^k\partial x^l}=0 \qquad ①$$

轮换指标 $j \to l \to k$

$$\sum_{i=1}^{3}\frac{\partial y^i}{\partial x^k\partial x^j}\frac{\partial y^i}{\partial x^l}+\sum_{i=1}^{3}\frac{\partial y^i}{\partial x^k}\frac{\partial^2 y^i}{\partial x^l\partial x^j}\equiv 0 \qquad ②$$

$$\sum_{i=1}^{3}\frac{\partial y^i}{\partial x^l\partial x^k}\cdot\frac{\partial y^i}{\partial x^j}+\sum_{i=1}^{3}\frac{\partial y^i}{\partial x^l}\cdot\frac{\partial^2 y^i}{\partial x^j\partial x^k}\equiv 0 \qquad ③$$

①+②−③: $$\sum\frac{\partial^2 y^i}{\partial x^j\partial x^l}\frac{\partial y^i}{\partial x^k}=0$$

由于 $\left(\frac{\partial y^i}{\partial x^k}\right)$ 正交, 非奇异 $\Longrightarrow \frac{\partial^2 y^i}{\partial x^j\partial x^l}\equiv 0$

$$y^i=y^i(x_1,x_2,x_3)=c_1x_1+c_2x_2+c_3x$$

观察特例 $\varphi:\mathbb{R}\to\mathbb{R}$ 保距

$\forall x, x_0\in\mathbb{R}$

$|x-x_0|=|\varphi(x)-\varphi(x_0)|$

$(x-x_0)^2=(\varphi(x)-\underset{\varphi_0}{\underbrace{\varphi(x_0)}})^2$

$(x-x_0)^2=(\varphi(x)-\varphi_0)^2$

对 x 求导: $(\varphi(x)-\varphi_0)\cdot\varphi'(x)=x-x_0 \qquad (2)$

再求导 $(\varphi'(x))^2+(\varphi(x)-\varphi_0)\varphi''(x)=1 \qquad (3)$

取 $x=x_0$, $(\varphi'(x_0))^2=1$

$\Downarrow$

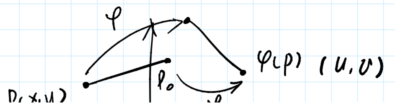
$\varphi'(x_0)=\pm 1$

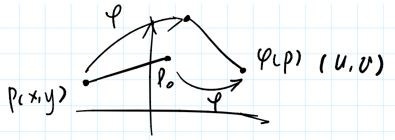
$\varphi'(x_0)=1 \Rightarrow y(x)=x+a$

$\varphi'(x_0)=-1 \Rightarrow y=-x+a$

二维 case

设 $\varphi: E^2\to E^2$ isometry





$$\varphi : \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

$$\varphi(p_0) : u_0 : u(x_0, y_0)$$
$$v_0 : v(x_0, y_0)$$

$$(x-x_0)^2 + (y-y_0)^2 = \left(u(x,y) - u_0\right)^2 + \left(v(x,y) - v_0\right)^2$$

对x求导

$$2(x-x_0) = 2\left(u(x,y) - u_0\right)\frac{\partial u}{\partial x}(x,y) + 2\left(v(x,y) - v_0\right)\frac{\partial v}{\partial x}(x,y)$$

对 y ··

···

$$(u-u_0)\frac{\partial u}{\partial x}(x,y) + (v-v_0)\frac{\partial v}{\partial x} = x - x_0 \qquad ①$$

$$(u-u_0)\frac{\partial u}{\partial y}(x,y) + (u-v_0)\frac{\partial v}{\partial y} = y - y_0 \qquad ②$$

① 再求偏导

$$\left(\frac{\partial u}{\partial x}\right)^2 + (u-u_0)\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 + (v-v_0)\frac{\partial^2 v}{\partial x^2} = 1$$

$$\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x} + (u-u_0)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial x} + (v-v_0)\frac{\partial^2 v}{\partial x \partial y} = 0$$

② 再求导

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + (u-u_0)\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + (v-v_0)\frac{\partial^2 v}{\partial y \partial x} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)^2 + (u-u_0)\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2 + (v-v_0)\frac{\partial^2 v}{\partial y^2} = 1$$

at $P_0(x_0, y_0)$

$$\left(\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 1$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = 1$$

$$\Longrightarrow \left.\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right|_{(x_0,y_0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$u = x + c$

$v = y + d$