Curvature & torsion

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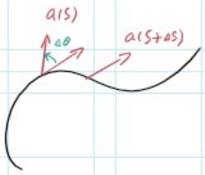
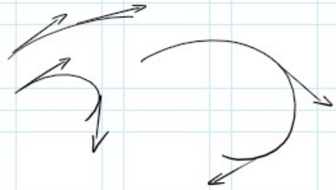
曲率和挠率的含义

r(s) 本身就是一个向量，单位化 r(s) 变成 a(s)？ $\forall s$

Prop 设曲线 $C: r=r(s)$ 上每一点都指定了一个单位向量 $a(s)$：$|a(s)|\equiv 1$.



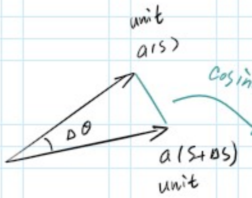


则 $$|\dot{a}(s)| = \lim_{\Delta s\to 0}\left|\frac{\Delta\theta}{\Delta s}\right|.$$

其中 $\Delta\theta$ 表示 $a(s)$ 与 $a(s+\Delta s)$ 的夹角

$$|\dot{a}(s)| = \lim_{\Delta s\to 0}\left|\frac{a(s+\Delta s)-a(s)}{\Delta s}\right| = \lim_{\Delta s\to 0}\frac{2\left|\sin\frac{\Delta\theta}{2}\right|}{|\Delta s|}$$

$$= \lim_{\Delta s\to 0}\frac{\left|\sin\frac{\Delta\theta}{2}\right|}{\left|\frac{\Delta\theta}{2}\right|}\cdot\left|\frac{\Delta\theta}{\Delta s}\right| = \lim_{\Delta s\to 0}\left|\frac{\Delta\theta}{\Delta s}\right|$$



Cosine theorem

$$|a(s+\Delta s)-a(s)| = (1^2+1^2-2\times 1\times 1\times\cos\Delta\theta)^{\frac{1}{2}}$$
$$= (2(1-\cos\Delta\theta))^{\frac{1}{2}} = \left(4\sin^2\frac{\Delta\theta}{2}\right)^{\frac{1}{2}} = \left|2\sin\frac{\Delta\theta}{2}\right|$$

一般参数的曲率、挠率公式. 结果对于具有非逗留点的正则曲线 $C: r=r(t)\ (t\in I)$
C 上任一点

$$\kappa = \frac{|r'\times r''|}{|r'|^3},\quad \tau = \frac{(r', r'', r''')}{|r'\times r''|^2}$$

Pf.
$$\begin{cases}\dot{T} = \kappa N\\ \dot{N} = -\kappa T + \tau B\\ \dot{B} = -\tau N\end{cases}$$

$$T = \frac{r'}{|r'|}\qquad B = \frac{r'\times r''}{|r'\times r''|},\quad N = B\times T$$

$s = \int_0^t |r'(u)|\,du$，令 $t=0$ 时 $s=0$.

$$\frac{ds}{dt} = |r'(t)|,\quad \frac{dt}{ds} = \frac{1}{|r'(t)|}$$

$$T = \frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = r'\frac{dt}{ds}$$

$$\dot{T} = \frac{d}{ds}\left(r'\frac{dt}{ds}\right) = \left(\frac{dr'}{dt}\cdot\frac{dt}{ds}\right)\frac{dt}{ds} + r'\frac{d}{ds}\left(\frac{dt}{ds}\right)$$
$$= r''\left(\frac{dt}{ds}\right)^2 + r'\frac{d^2t}{ds^2}$$

$$\dot{T} = \kappa N$$

$$\dot{T}\times T = \kappa N\times T = -\kappa B$$

取模 $|\dot{T}\times T| = \kappa$

$$\dot{T}\times T = \left[r''\left(\frac{dt}{ds}\right)^2 + r'\frac{d^2t}{ds^2}\right]\times\left(\frac{r'}{|r'|}\right)$$

$$= \frac{r''\times r'}{|r'|}\left(\frac{dt}{ds}\right)^2 \quad 取模\ \kappa = \frac{|r'\times r''|}{\ }$$

$$= \frac{r'' \times r'}{|r'|}\left(\frac{dt}{ds}\right)^2 \quad \text{取模} \quad k = \frac{|r' \times r''|}{|r'|^3}$$

$$\frac{r'' \times r'}{|r'| \underbrace{\left(\frac{ds}{dt}\right)^2}_{= |r'|^2}}$$

torsion: $\tau = -\dot{B} \cdot N = -\frac{d}{ds}(B) \cdot (B \times T)$

$$= \frac{d}{ds}(B) \cdot (T \times B)$$

$$= \frac{d}{ds}\left(\frac{r'(t) \times r''(t)}{|r' \times r''|}\right) \cdot \left[\frac{r'}{|r'|} \times \frac{r' \times r''}{|r' \times r''|}\right]$$

$$= \frac{d}{dt}\left(\frac{r' \times r''}{|r' \times r''|}\right)\left(\frac{dt}{ds}\right) \frac{1}{|r'||r' \times r''|} \cdot \left[-|r'|^2 r'' + (r' \cdot r'') r'\right]$$

$$= \frac{1}{|r'|^2} \frac{1}{|r' \times r''|}\left[\frac{d}{dt}\left(\frac{1}{|r' \times r''|}\right) r' \times r'' + \frac{1}{|r' \times r''|} r' \times r'''\right] \cdot \left[-|r'|^2 r'' + (r' \cdot r'') r'\right]$$

$$= \frac{1}{|r'|^2 |r' \times r''|} \frac{r' \times r'''}{|r' \times r''|}\left[-|r'|^2 r'' + (r' \cdot r'') r'\right]$$

$$= \frac{-1}{|r' \times r''|^2}(r', r''', r'') + 0 = \frac{(r', r'', r''')}{|r' \times r''|^2}$$

挠率の几何意义

Recall: Frenet formula

$$\begin{cases} \dot{T} = kN \\ \dot{N} = -kT + \tau B \\ \dot{B} = -\tau N \end{cases}$$

可让我们对曲线的一般性质(局部)有比较深入的了解

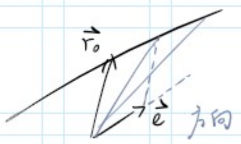
(注: 当曲线以弧长为参数)

<u>命题 1</u> 曲线为直线 $\Longleftrightarrow k \equiv 0$

Pf. C为直线, 则C的方程可写为 $\vec{r}(t) - \vec{r}_0 = t\vec{e}$ $\quad \vec{e}$ 为单位向量

$\vec{r}'(t) = \vec{e}$

$|r'(t)| = 1$, 此时 t 为弧长参数



recall: 方向导数的定义

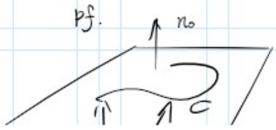
$\dot{T}(s) \equiv 0 \Rightarrow k \equiv 0$

若 $k \equiv 0$, 则 $\dot{T}(s) \equiv 0$, 则 $T(s) \equiv e$ (常向量)

$\frac{dr}{dt} = e$, $\quad r(t)$ 直线

<u>命题 2</u> C为平面曲线(非直线) $\Longleftrightarrow \tau \equiv 0$

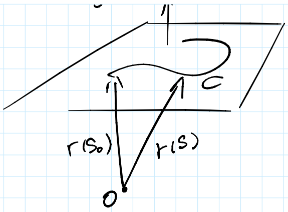
Pf.



$(\Rightarrow)$ C: $r = r(s)$ 为平面曲线

$(r(s) - r(s_0)) \cdot n_0 = 0$

$n_0 \neq 0$ 为 const vector



$(r(s)-r(s_0))\cdot n_0=0$

$n_0\neq 0$ 为 Const vector

求导 on both sides， $T(s)\cdot n_0=0$ ②

再求导 on s， $kN\cdot n_0=0$ ③

由于 $k\neq 0$， $N(s)\cdot n_0\equiv 0$ ③'

再对 s 求导 $(-kT+\tau B)\cdot n_0=0$ ④

$\tau(B\cdot n_0)\equiv 0$

if $\tau\neq 0$ then $B\cdot n_0=0$ ⑤

②③③'⑤ 得 $n_0=0$ 与假设矛盾. 故 $\tau=0$

$(\Leftarrow)$ $\tau=\dfrac{(r',r'',r''')}{|r'\times r''|^2}\equiv 0 \Rightarrow (r',r'',r''')\equiv 0$

$\dot{B}(s)=-\tau N\equiv 0$

则 $B(s)=B_0$ （常向量）

$T(s)\cdot B(s)\equiv 0\Rightarrow T(s)\cdot B_0\equiv 0$

$\frac{d}{ds}(r(s)\cdot B_0)\equiv 0 \Rightarrow r(s)\cdot B_0=$ 常数

局部规范形式 $\longrightarrow$ τ 的意义

对空间曲线 $C: r=r(s)$ 在 P_0 附近. 看曲线的形状.

以 P_0 为弧长起点 $(s=0)$，且以 P_0 处 Frenet 标架 $\Sigma=\{r(0),T(0),N(0),B(0)\}$

作为新的坐标系. 观察曲线 C 在 $\{r(0),T(0),N(0),B(0)\}$ 下看 $r(s)-r(s_0)$

利用 Taylor 展开

$$r(s)-r(s_0)=\dot{r}(s_0)s+\frac{\ddot{r}(s_0)}{2!}s^2+\frac{\dddot{r}(s_0)}{3!}s^3+\varepsilon s^3,\ \text{where}\ \lim_{s\to 0}\varepsilon=0$$

注意, $\dot{r}(s)=T(s)$， $\ddot{r}(s)=\dot{T}(s)=kN(s)$

$\dddot{r}(s)=(kN)'=\dot{k}N+k(-kT+\tau B)$

$$r(s)-r(s_0)=sT_0+\frac{1}{2}s^2k_0N(0)+\frac{s^3}{6}\left(-k_0^2T(0)-\dot{k}_0N(0)+k_0\tau_0B(0)\right)+s^3\varepsilon \quad 记\ \varepsilon=\varepsilon_1\vec{T}(0)+\varepsilon_2\vec{N}(0)+\varepsilon_3\vec{B}(0)$$

$$=\left(s-\frac{k_0}{6}s^3+\varepsilon_1\right)\vec{T}(0)+\left(\frac{k_0}{2}s^2-\frac{k_0^2}{6}s^3+\varepsilon_2\right)\vec{N}(0)+\left(\frac{1}{6}k_0\tau_0+\varepsilon_3\right)\vec{B}(0)$$

$$\begin{cases} x=s-\frac{k_0}{6}s^3+\varepsilon_1s^3 \\ y=\frac{1}{2}k_0s^2-\frac{1}{6}\dot{k}_0s^3+\varepsilon_2s^3 \\ z=\frac{1}{6}k_0\tau_0s^3+\varepsilon_3s^3 \end{cases}$$

考虑曲线

$$\bar{C}\quad\begin{cases} x=s \\ y=\frac{1}{2}k_0s^2 \\ z=\frac{1}{6}k_0\tau_0s^3 \end{cases}$$

称为 C 在 P_0 处的近似曲线