Frenet 公式

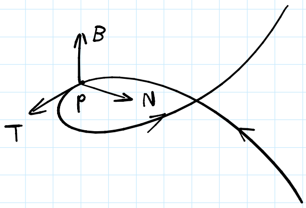
2020年9月24日 13:45

3-d case

Joseph-Alfred Serret 1819 ~ 1885

Frédéric-Jean Frenét 1816 ~ 1900

Frenet-Serret 公式



对于曲线 C: $r = r(s)$ (S 弧长)

C 每一点处的 Frenet 标架 $\Sigma = \{r(s); T(s), N(s), B(s)\}$

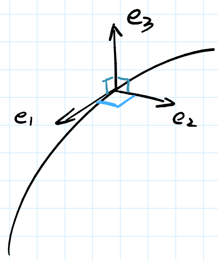
考察沿 C 标架关于弧长的"变化率" 即看 $\dot{T}(s)$, $\dot{N}(s)$, $\dot{B}(s)$

以 $\{r(s); T(s), N(s), B(s)\}$ 为参照系，$\dot{T}, \dot{N}, \dot{B}$ 可用 T, N, B 线性表示：

$$\begin{cases} \dot{T}(s) = 0\, T + k N + 0\, B \\ \dot{N}(s) = -k T + 0 \cdot N + \tau B \\ \dot{B}(s) = 0 \cdot T - \tau N + 0 \cdot B \end{cases}$$

(这个式子待会证)

Lemma 假设曲线 C: $r = r(t)$, $t \in I$



每一点处都指定了一组 orthonormal 向量 $(e_1(t), e_2(t), e_3(t))$

其中 $\langle e_i, e_j \rangle = \delta_{ij}$.

i.e., $\langle e_i, e_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$. 注意 e_i depends on t!

differentiate both sides over t:

$$(e_i \cdot e_j)' = e_i' \cdot e_j + e_i \cdot e_j' = 0 \quad \forall t$$

又有 $e_i' \in E^3$, $i = 1, 2, 3$, 则 e_i' can be written as a linear combination of e_1, e_2, e_3.

3-d Euclidean space

$$e_1' = a_{11} e_1 + a_{12} e_2 + a_{13} e_3$$

$$e_1' \cdot e_1 = a_{11} + 0 + 0$$

$$e_1' \cdot e_2 = 0 + a_{12} + 0$$

$$e_1' \cdot e_3 = 0 + 0 + a_{13}$$

More generally, $e_i' \cdot e_j = a_{ij}$, $e_i \cdot e_j' = a_{ji}$

$$\Downarrow \quad e_i' \cdot e_j + e_i \cdot e_j' = 0$$

$$a_{ij} + a_{ji} = 0 \qquad \text{skew-symmetric!}$$

$$\Downarrow$$

$$a_{ij} + a_{ji} = 0 \qquad \text{skew-symmetric !}$$

Recall: Frenet frame and $r(s)$

$$T = \dot{r}(s), \quad N = \frac{\ddot{r}(s)}{|\ddot{r}(s)|}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{\dot{T} = \ddot{r}(s) \qquad \ddot{r}(s) = |\ddot{r}(s)|\, N}$$

$$\Downarrow$$

$$\dot{T} = |\ddot{r}(s)|\, N$$

记 $\kappa(s) = |\ddot{r}(s)| > 0$，则

$$\dot{T} = 0T + \kappa(s) N + 0B$$

(由skew-symmetry) $$\dot{N} = -\kappa(s) T + 0N + a_{23} B$$

$$\dot{B} = 0T + (-a_{23}) N + a_{33} B$$

记 $a_{23} = \tau$，则 $\dot{N} = -\kappa T + \tau B$. Remember that κ and τ depend on t !

find a_{33}

$$B = T \times N = \dot{r}(s) \times \frac{\ddot{r}(s)}{|\ddot{r}(s)|}$$

$$\dot{B} = \frac{1}{|\ddot{r}(s)|}\left(\cancel{\ddot{r}(s) \times \ddot{r}(s)} + \dot{r}(s) \times \dddot{r}(s)\right) = -\tau \frac{\ddot{r}(s)}{|\ddot{r}(s)|} + \frac{a_{33}}{|\ddot{r}(s)|}\left(\dot{r}(s) \times \ddot{r}(s)\right)$$

$$\dot{r}(s) \times \dddot{r}(s) = -\tau \ddot{r}(s) + a_{33}\, \dot{r}(s) \times \ddot{r}(s)$$

公式 $$\begin{cases} \dot{T} = \kappa N \\ \dot{N} = -\kappa T + \tau B \\ \dot{B} = -\tau N \end{cases}$$ 称为曲线C的Frenet-Serret公式

$$\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

问：这组公式告诉我们什么？

κ与τ是C的几何量

def 称 $\kappa = |\ddot{r}(s)|$ 为空间曲线C在$r(s)$处的曲率 (curvature)

$\tau = -\dot{B} \cdot N = B \cdot \dot{N}$ 为C的挠率 (torsion)

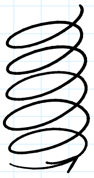
(Next Page)

$\tau = -\dot{B}\cdot N = B\cdot\dot{N}$ 为 C 的挠率 (torsion)

(Next page)

例1.(圆周)

例2.(圆柱螺线)



$$r(t) = (a\cos t,\ a\sin t,\ bt),\quad a>0$$

$$r'(t) = (-a\sin t,\ a\cos t,\ b)$$

$$|r'(t)| = (a^2+b^2)^{\frac{1}{2}}$$

$$S(t) = \int_0^t (a^2+b^2)^{\frac{1}{2}}dt = \sqrt{a^2+b^2}\,t$$

从而 $r = r(s) = \left(a\cos\frac{S}{\sqrt{a^2+b^2}},\ a\sin\frac{S}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}}S\right)$

$$\dot{r}(S) = \left(\frac{-a}{\sqrt{a^2+b^2}}\sin\frac{S}{\sqrt{a^2+b^2}},\ \frac{a}{\sqrt{a^2+b^2}}\cos\frac{S}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}}\right)$$

$\parallel$

T

$$\dot{T}(S) = \left(-\frac{a}{a^2+b^2}\cos\frac{S}{\sqrt{a^2+b^2}},\ -\frac{a}{a^2+b^2}\sin\frac{S}{\sqrt{a^2+b^2}},\ 0\right)$$

$$= \frac{a}{a^2+b^2}\left(-\cos\frac{S}{\sqrt{a^2+b^2}},\ -\sin\frac{S}{\sqrt{a^2+b^2}},\ 0\right)$$

与 $\dot{T}(S) = k N(S)$ 比较 $\Rightarrow k = \frac{a}{a^2+b^2} > 0$

$$N(S) = \left(-\cos\frac{S}{\sqrt{a^2+b^2}},\ -\sin\frac{S}{\sqrt{a^2+b^2}},\ 0\right)$$

$$T = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b),\quad N = (-\cos t, -\sin t, 0)$$

$$T\times N = \frac{1}{\sqrt{a^2+b^2}}(b\sin t,\ -b\cos t,\ a) = \frac{1}{\sqrt{a^2+b^2}}\left(b\sin\frac{S}{\sqrt{a^2+b^2}},\ -b\cos\frac{S}{\sqrt{a^2+b}},\ a\right)$$

$$B = \frac{b}{a^2+b^2}\left(\cos\frac{S}{\sqrt{a^2+b^2}},\ \sin\frac{S}{\sqrt{a^2+b^2}},\ 0\right)$$

$\parallel$

$-\tau\vec{N} \Rightarrow \tau = \frac{b}{a^2+b^2}$

符号？

$a+b$

符号 ?