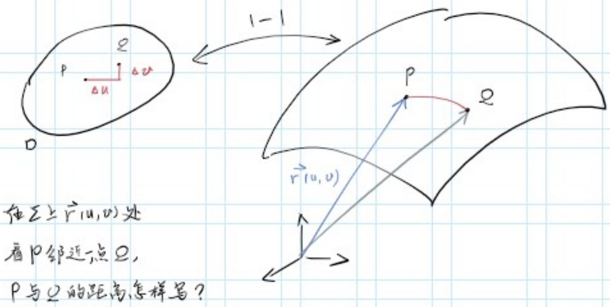
第一基本形式

2020年10月13日 15:14

曲面的第一基本形式 E F G

从最简单，最基本的"动作"问题开始。假设给定正则曲面

$$\Sigma: \vec{r} = \vec{r}(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v)\big), \quad (u,v) \in D$$



在 Σ 上 $\vec{r}(u,v)$ 处

看 P 邻近一点 Q，

P 与 Q 的距离怎样写？

设 Q 对应参数 $(u+\Delta u,\ v+\Delta v)$

于是 $\vec{r}(Q) = \big(x(u+\Delta u, v+\Delta v),\ y(\ ),\ z(\ )\big)$

$\qquad = \vec{r}(u+\Delta u, v+\Delta v)$

$\vec{r}(P) = \vec{r}(u,v)$

$$|\vec{PQ}|^2 = |\vec{r}(u+\Delta u, v+\Delta v) - \vec{r}(u,v)|^2$$

since $\vec{r}(u+\Delta u, v+\Delta v) - \vec{r}(u,v)$

(Taylor) $= \vec{r}_u(u,v)\Delta u + \vec{r}_v(u,v)\Delta v + \vec{\varepsilon}$,

$$|\vec{r}(u+\Delta u, v+\Delta v) - \vec{r}(u,v)|^2 = \big(\vec{r}_u(u,v)\Delta u + \vec{r}_v(u,v)\Delta v + \vec{\varepsilon}\big)\cdot\big(\vec{r}_u(u,v)\Delta u + \vec{r}_v(u,v)\Delta v + \vec{\varepsilon}\big)$$

$$= (\vec{r}_u\cdot\vec{r}_u)(\Delta u)^2 + 2(\vec{r}_u\cdot\vec{r}_v)\Delta u\Delta v + (\vec{r}_v\cdot\vec{r}_v)(\Delta v)^2 + \text{高阶无穷小}$$

注意 $\vec{r}_u(u,v)$, $\vec{r}_v(u,v)$ 与 $\Delta u, \Delta v$ 无关

记 $E = \vec{r}_u\cdot\vec{r}_u$, $F = \vec{r}_u\cdot\vec{r}_v$, $G = \vec{r}_v\cdot\vec{r}_v$

$$|\vec{PQ}| \approx E(\Delta u)^2 + 2F\Delta u\Delta v + G(\Delta v)^2$$

分析上，对于自变量 u 和 v，将 Δu 和 Δv 记 du, dv resp.

那么 $|\vec{PQ}| \approx E(du)^2 + 2F\,du\,dv + G(dv)^2$ 是一个二次齐次

def 对于正则曲面 $\Sigma: \vec{r} = \vec{r}(u,v) = \big(x(\ ), y(\ ), z(\ )\big)$, $(u,v)\in D$,

令 $E = \vec{r}_u\cdot\vec{r}_u$, $F = \vec{r}_u\cdot\vec{r}_v$, $G = \vec{r}_v\cdot\vec{r}_v$

称关于微分 du, dv 的二次型

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

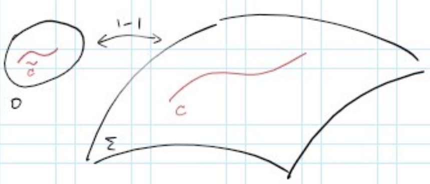
为 Σ 在 P 处的第一基本形式

$$I = (du\ \ dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

注. 从另一个基本问题也会引申出第一基本形式

原因. 对于曲面 $\Sigma: \vec{r}(u,v)$

取 Σ 上一条曲线 C



$\tilde{C}$: $u = u(t)$, $v = v(t)$ $\quad a \le t \le b$

C: $\vec{r} = \vec{r}(u(t), v(t))$

$v = v(t)$

$C: \vec{r} = \vec{r}(u(t), v(t))$

C 的长度 $s = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^t \sqrt{E\left(\frac{du}{d\tau}\right)^2 + 2F\frac{du}{d\tau}\frac{dv}{d\tau} + G\left(\frac{dv}{d\tau}\right)^2}\, d\tau$

Since $\frac{d\vec{r}}{dt} = \vec{r}_u \frac{du}{dt} + \vec{r}_v \frac{dv}{dt}$

$$\left|\frac{d\vec{r}}{dt}\right|^2 = \left(\vec{r}_u \frac{du}{dt} + \vec{r}_v \frac{dv}{dt}\right)(\qquad)$$

$$= E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

$$\frac{ds}{dt} = \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}$$

曲线C 弧长微分 $ds = \sqrt{E(du)^2 + 2F\,du\,dv + G(dv)^2}$

$$\Rightarrow ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

因此, 今后也常常将曲面第一基本形式记作 ds^2

E, F, G 称为曲面的第一基本量.

注. $\vec{r} = \vec{r}(u, v) = (\qquad)$

$$d\vec{r} = (dx, dy, dz)$$

$$= (x_u du + x_v dv,\ y_u du + y_v dv,\ z_u du + z_v dv)$$

$$= (x_u, y_u, z_u)\,du + (x_v, y_v, z_v)\,dv$$

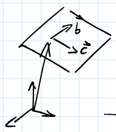
$$d\vec{r} = \vec{r}_u\,du + \vec{r}_v\,dv$$

$$d\vec{r}\cdot d\vec{r} = E(du)^2 + 2F\,du\,dv + G(dv)^2 = I$$

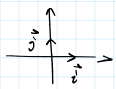
$$I = d\vec{r}\cdot d\vec{r}$$

examples

(1) 平面.



$\vec{r} = \vec{r}_0 + u\vec{a} + v\vec{b}$



$\vec{r} = \vec{r}(x, y) = (x, y, 0)$ $\quad d\vec{r} = (dx, dy, 0)$

$\vec{r}_x = (1, 0, 0)$ $\quad I = d\vec{r}\cdot d\vec{r} = (dx)^2 + (dy)^2$

$\vec{r}_y = (0, 1, 0)$

$E = 1,\ F = 0,\ G = 1$

$$\Rightarrow I = (dx)^2 + (dy)^2 = (dx\ \ dy)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}$$

极坐标下第一基本形式 $\vec{r} = (x, y, z)$

$= (\rho\cos\theta, \rho\sin\theta, 0)$ $\quad d\vec{r} = (\cos\theta\,d\rho + \rho\sin\theta\,d\theta,\ \sin\theta\,d\rho + \rho\cos\theta\,d\theta,\ 0)$

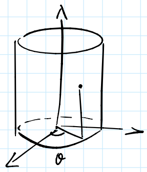
$\vec{r}_\rho = (\cos\theta, \sin\theta, 0)$ $\quad I = d\vec{r}\cdot d\vec{r} = \cdots$

$\vec{r}_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$

$E = 1,\ F = 0,\ G = \rho^2$

$$I = (d\rho)^2 + \rho^2(d\theta)^2 = (d\rho\ \ d\theta)\begin{pmatrix}1 & 0\\ 0 & \rho^2\end{pmatrix}\begin{pmatrix}d\rho\\ d\theta\end{pmatrix}$$

2) cylinder.



$\Sigma: \vec{r} = \vec{r}(\theta, z)$

$= (a\cos\theta, a\sin\theta, z$

$\vec{r}_\theta = (-a\sin\theta, a\cos\theta, 0)$

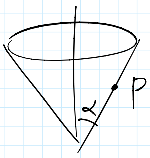
$\vec{r}_z = (0, 0, 1)$

$E = a^2,\ F = 0,\ G = 1$

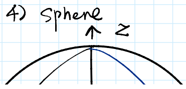
$I = a^2(d\theta)^2 + (dz)^2$

3) cone

(HW)



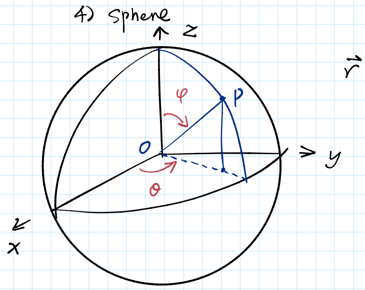
4) sphere

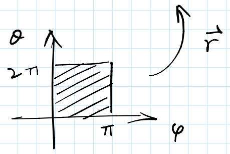


$\vec{r} = (x, y, z) = (a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi)$

)

4) sphere





$$\vec{r} = (x, y, z) = (a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi)$$

$$0 \le \theta < 2\pi$$

$$0 \le \varphi \le \pi$$

$$\vec{r}_\varphi = (a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi)$$

$$\vec{r}_\theta = (-a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0)$$

$$E = \vec{r}_\varphi \cdot \vec{r}_\varphi = a^2.\quad F = 0,\quad G = a^2\sin^2\varphi$$

$$I = a^2(d\varphi)^2 + a^2\sin^2\varphi\, d\varphi\, d\theta$$

$$= a^2\left((d\varphi)^2 + \sin^2\varphi\,(d\theta)^2\right)$$

第一基本形式有什么用?

曲面上涉及长度, 面积, 夹角的几何量都能由第一基本形式表达