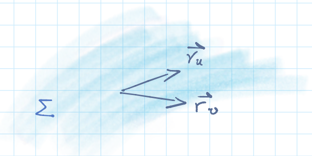
Weingarten矩阵

2020年10月28日 16:30



Weingarten 变换

在自然基$\{\vec{r_u}, \vec{r_v}\}$下的矩阵.

设 $\vec{n_u} = x\vec{r_u} + y\vec{r_v}$

$$-\vec{r_u}\cdot\vec{n_u} = xE + yF = L \qquad -\vec{r_v}\cdot\vec{n_u} = xF + yG = M \qquad \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L \\ M \end{pmatrix}$$

$$\begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2 = (\vec{r_u}\cdot\vec{r_v})\cdot(\vec{r_u}\cdot\vec{r_v}) > 0$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L \\ M \end{pmatrix}$$

$-\vec{n_v} = x\vec{r_u} + y\vec{r_v}$ (不是上面的 x, y)

$$M = xE + yF \qquad N = xF + yG \qquad \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix}$$

$$-\vec{n_u} = (\vec{r_u} \ \ \vec{r_v})\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L \\ M \end{pmatrix} := \begin{pmatrix} x \\ y \end{pmatrix}$$

$$-\vec{n_v} = (\vec{r_u} \ \ \vec{r_v})\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} M \\ N \end{pmatrix} := \begin{pmatrix} x' \\ y' \end{pmatrix}$$

记 $(-\vec{n_u} \ \ -\vec{n_v}) = (\vec{r_u} \ \ \vec{r_v})\, A$ concatenate

即 $$(W(\vec{r_u}), W(\vec{r_v})) = (\vec{r_u} \ \ \vec{r_v})\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

Weingarten 变换在自然基$\{\vec{r_u}, \vec{r_v}\}$下的矩阵为

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

A 的 eigen's.

特征多项式 $f(\lambda) = |\lambda I - A|$

$$= \left| \lambda I - \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} \right|$$

$$= \det\left(\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\left(\lambda \begin{bmatrix} E & F \\ F & G \end{bmatrix} - \begin{bmatrix} L & M \\ M & N \end{bmatrix}\right)\right)$$

$$= \frac{1}{EG-F^2}\det\begin{pmatrix} \lambda E - L & \lambda F - M \\ \lambda F - M & \lambda G - N \end{pmatrix}$$

$$= \frac{1}{EG-F^2}\left(\lambda^2(EG-F^2) + \lambda(2FM - EN - LG) + LN - M^2\right)$$

令 $f(\lambda) = 0$ i.e., $\lambda^2(EG-F^2) + \lambda(2FM - EN - LG) + LN - M^2 = 0$ 只要记 $\det\left(\lambda\begin{pmatrix} E & F \\ F & G \end{pmatrix} - \begin{pmatrix} L & M \\ M & N \end{pmatrix}\right) = 0$

$$\Rightarrow \lambda_1 + \lambda_2 = \frac{LG + EN - 2FM}{EG - F^2}$$

$$\lambda_1\lambda_2 = \frac{LN - M^2}{EG - F^2}$$

$$\Rightarrow \text{平均曲率 } H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\,\frac{LG + EN - 2FM}{EG - F^2}$$

Gauss 曲率 $K = k_1 k_2 = \dfrac{LN - M^2}{EG - F^2}$