Weingarten变换

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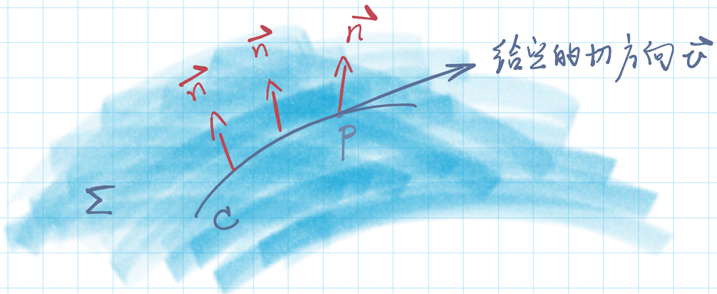
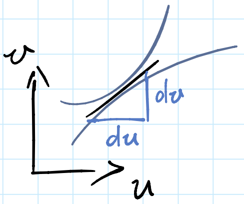


Consider : $\left.\dfrac{d\vec n}{dt}\right|_{t=0}$

给定一个方向 $\vec v$，可以选一条过 P 且沿 $\vec v$ 的曲线 C.



方向可用 (du, dv) 表示. 沿 $\vec v$：C在P的切线方向与 $\vec v$ 相同.

设C $\begin{cases} u=u(t) \\ v=v(t) \end{cases}$, $t=0$ 对应P点.

C: $\vec r=\vec r(t)=\vec r(u(t),\, v(t))$

$$C\text{在P处切线 } \vec v = \left.\frac{d\vec r}{dt}\right|_{t=0} = \vec r_u(u_0,v_0)\left.\frac{du}{dt}\right|_{t=0} + \vec r_v(u_0,v_0)\left.\frac{dv}{dt}\right|_{t=0}$$

$$:= \lambda \vec r_u + \mu \vec r_v$$

沿C, $\vec n=\vec n(u,v)=\vec n(u(t),\, v(t))$

$$\left.\frac{d\vec n}{dt}\right|_{t=0} = \vec n_u(u_0,v_0)\left.\frac{du}{dt}\right|_{t=0} + \vec n_v(u_0,v_0)\left.\frac{dv}{dt}\right|_{t=0}$$

$$= \lambda \vec n_u + \mu \vec n_v$$

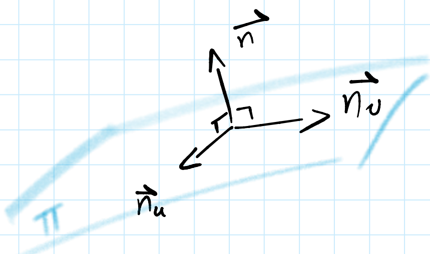
$\vec n_u$, $\vec n_v$ lies in the tangent plane !!!

unit => $\vec n\cdot\vec n=1$, i.e. $\vec n(u,v)\cdot\vec n(u,v)=1$

求导 over u : $\vec n_u\cdot\vec n+\vec n\cdot\vec n_u=0$

$\vec n_u\cdot\vec n=0$

求导 over v : $\vec n_v\cdot\vec n=0$



求导 over v: $\vec{n}_v \cdot \vec{n} = 0$

我们得到 Π 中的一个 operater $\sigma: \Pi \to \Pi$

$$\vec{v} = \lambda \vec{r}_u + \mu \vec{r}_v \longmapsto \lambda \vec{n}_u + \mu \vec{n}_v$$

取 $\lambda=1, \mu=0$: $\vec{r}_u \mapsto \vec{n}_u$

$\lambda=0, \mu=1$: $\vec{r}_v \mapsto \vec{n}_v$ (Since 一开始方向 $\vec{v}$ 是任取的)

σ 是 operator since σ is linear.

Weingarten 变换

<u>def</u> 切平面上一个变换 W:

$$\vec{v} = \lambda \vec{r}_u + \mu \vec{r}_v \longmapsto -\lambda \vec{n}_u - \mu \vec{n}_v$$

称为 Σ 在 P 点处的 weingarten 变换

Some basic Properties

(1) Weingarten 变换是对称变换:

$$\langle W(v), w \rangle = \langle v, W(w) \rangle$$

Pf. 设 $v = a_1 \vec{r}_u + a_2 \vec{r}_v$, $w = b_1 \vec{r}_u + b_2 \vec{r}_v$

$$\langle W(v), w \rangle = \langle -a_1 \vec{n}_u - a_2 \vec{n}_v, b_1 \vec{r}_u + b_2 \vec{r}_v \rangle$$
$$\langle v, W(w) \rangle = \langle a_1 \vec{r}_u + a_2 \vec{r}_v, -b_1 \vec{n}_u - b_2 \vec{n}_v \rangle$$

系数换一下还是相等的. □

(2) W 变换能表达法曲率和第二基本形式

$$\begin{aligned} W(d\vec{r}) &= W(\vec{r}_u du + \vec{r}_v dv) \\ &= W(\vec{r}_u) du + W(\vec{r}_v) dv \\ &= -\vec{n}_u du - \vec{n}_v dv \\ &= -d\vec{n} \end{aligned}$$

$= -d\vec{n}$

$\Rightarrow$ $\mathrm{II} = -d\vec{n} \cdot d\vec{r}$ 负号出现. (Review Page: 第二基本形式)

$= \langle \mathcal{W}(d\vec{r}), d\vec{r} \rangle$

$\Rightarrow$ $k_n = \dfrac{\mathrm{II}}{\mathrm{I}} = \dfrac{\langle \mathcal{W}(d\vec{r}), d\vec{r} \rangle}{\langle d\vec{r}, d\vec{r} \rangle}$